2-5 The Calculus of Scalar and Vector Fields (pp.33-55)

Fields are **functions** of coordinate variables (e.g., x, ρ, θ)

Q: How can we integrate or differentiate **vector** fields ??

A: There are many ways, we will study:

1. 4.

2. 5.

3. 6.

A. The *Integration* of Scalar and Vector Fields

1. *The Line Integral*

$$\int_C \mathbf{A}(\bar{r}_c) \cdot \overline{d\ell}$$

Q1:

A1:

- HO: Differential Displacement Vectors
- HO: The Differential Displacement Vectors for Coordinate Systems

Q2:

A2: HO: The Line Ingtegral

Q3:

A3:

- HO: The Contour C
- HO: Line Integrals with Complex Contours

Q4:

A4: HO: Steps for Analyzing Line Integrals

Example: The Line Integral

2. *The Surface Integral*

Another important integration is the **surface integral**:

$$\iint_S \mathbf{A}(\bar{r}_s) \cdot \overline{ds}$$

Q1:

A1:

HO: Differential Surface Vectors

HO: The Differential Surface Vectors for Coordinate Systems

Q2:

A2: HO: The Surface Integral

Q3:

A3:
- HO: The Surface S
- HO: Integrals with Complex Surfaces

Q4:

A4:
- HO: Steps for Analyzing Surface Integrals
- Example: The Surface Integral

3. *The Volume Integral*

The third important integration is the **volume integral**—it's the easiest of the 3!

$$\iiint_V g(\bar{r})\, dv$$

Q1:

A1:

HO: The Differential Volume Element

HO: The Volume V

Example: The Volume Integral

B. The *Differentiation* of Vector Fields

1. The Gradient

The **Gradient** of a **scalar** field $g(\bar{r})$ is expressed as:

$$\nabla g(\bar{r})$$

$$\nabla g(\bar{r}) = \mathbf{A}(\bar{r})$$

Q:

A: HO: The Gradient

Q:

A: HO: The Gradient Operator in Coordinate Systems

Q: The gradient of every scalar field is a vector field—does this mean every vector field is the gradient of some scalar field?

A:

HO: The Conservative Field

Example: Integrating the Conservative Field

2. Divergence

The **Divergence** of a vector field $\mathbf{A}(\bar{r})$ is denoted as:

$$\nabla \cdot \mathbf{A}(\bar{r})$$

$$\nabla \cdot \mathbf{A}(\bar{r}) = g(\bar{r})$$

Q:

A: HO: The Divergence of a Vector Field

Q:

A: HO: The Divergence Operator in Coordinate Systems

HO: The Divergence Theorem

3. Curl

The **Curl** of a vector field $\mathbf{A}(\bar{r})$ is denoted as:

$$\nabla \times \mathbf{A}(\bar{r})$$

$$\nabla \times \mathbf{A}(\bar{r}) = \mathbf{B}(\bar{r})$$

Q:

A: HO: The Curl

Q:

A: HO: The Curl Operator in Coordinate Systems

HO: Stoke's Theorem

HO: The Curl of a Conservative Vector Field

4. The Laplacian

$$\nabla^2 g(\bar{r})$$

C. Helmholtz's Theorems

$$\nabla \cdot \mathbf{A}(\bar{r}) \qquad \text{and/or} \qquad \nabla \times \mathbf{A}(\bar{r})$$

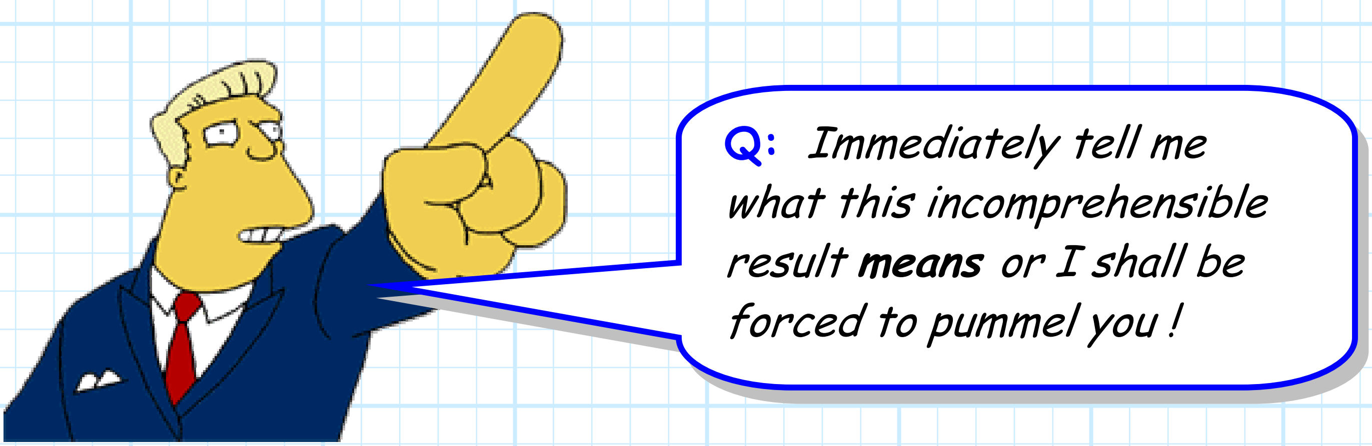
Q:

A: HO: Helmholtz's Theorems

Differential Displacement Vectors

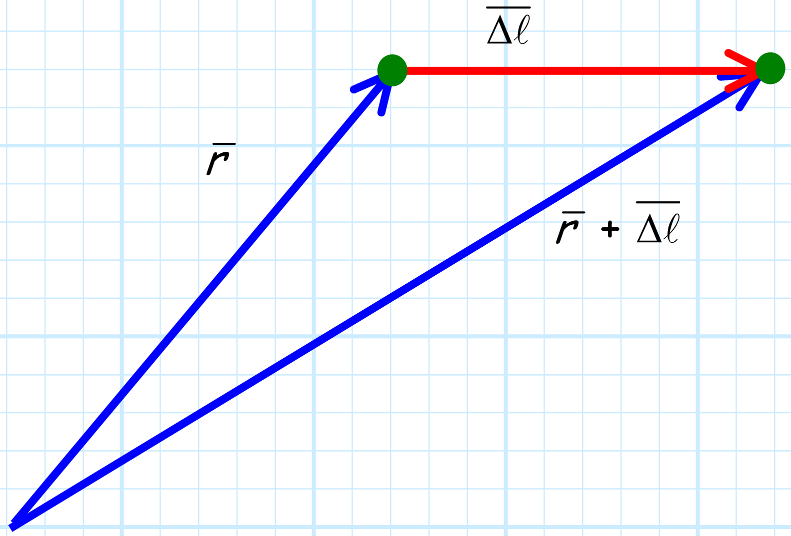
The derivative of a position vector $\bar{r}$, with respect to coordinate value ℓ (where $\ell \in \{x, y, z, \rho, \phi, r, \theta\}$) is expressed as:

$$
\begin{aligned}
\frac{d\bar{r}}{d\ell} &= \frac{d}{d\ell}\left(x\,\hat{a}_x + y\,\hat{a}_y + z\,\hat{a}_z\right) \\
&= \frac{d\left(x\,\hat{a}_x\right)}{d\ell} + \frac{d\left(y\,\hat{a}_y\right)}{d\ell} + \frac{d\left(z\,\hat{a}_z\right)}{d\ell} \\
&= \left(\frac{dx}{d\ell}\right)\hat{a}_x + \left(\frac{dy}{d\ell}\right)\hat{a}_y + \left(\frac{dz}{d\ell}\right)\hat{a}_z
\end{aligned}
$$



A: The vector above describes the **change** in **position vector** $\bar{r}$ due to a change in coordinate variable ℓ. This change in position vector is itself a vector, with both a **magnitude** and **direction**.

For example, if a point moves such that its coordinate ℓ changes from ℓ to $\ell + \Delta\ell$, then the position vector that describes that point changes from $\bar{r}$ to $\bar{r} + \overline{\Delta\ell}$.



In other words, this small vector $\overline{\Delta\ell}$ is simply a **directed distance** between the point at coordinate ℓ and its new location at coordinate $\ell + \Delta\ell$!

This directed distance $\overline{\Delta\ell}$ is related to the position vector derivative as:

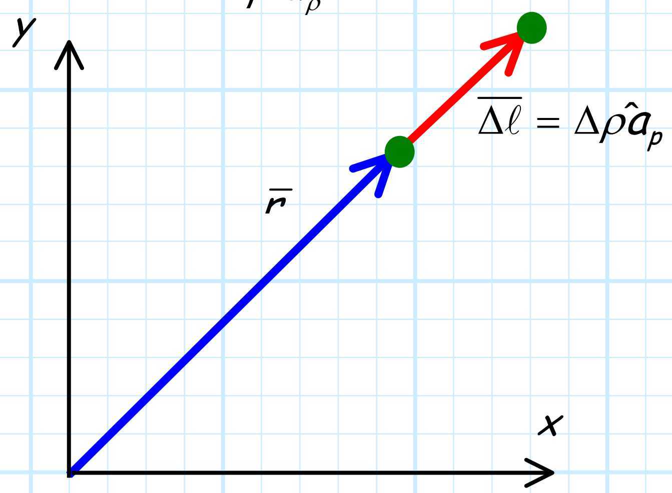
$$\begin{aligned}\overline{\Delta\ell} &= \Delta\ell \, \frac{d\,\bar{r}}{d\ell} \\ &= \Delta\ell\left(\frac{dx}{d\ell}\right)\hat{a}_x + \Delta\ell\left(\frac{dy}{d\ell}\right)\hat{a}_y + \Delta\ell\left(\frac{dz}{d\ell}\right)\hat{a}_z\end{aligned}$$

As an **example**, consider the case when $\ell = \rho$. Since $x = \rho\cos\phi$ and $y = \rho\sin\phi$ we find that:

$$
\begin{aligned}
\frac{d\,\overline{r}}{d\rho} &= \frac{dx}{d\rho}\,\hat{a}_x + \frac{dy}{d\rho}\,\hat{a}_y + \frac{dz}{d\rho}\,\hat{a}_z \\
&= \frac{d(\rho\cos\phi)}{d\rho}\,\hat{a}_x + \frac{d(\rho\sin\phi)}{d\rho}\,\hat{a}_y + \frac{dz}{d\rho}\,\hat{a}_z \\
&= \cos\phi\,\hat{a}_x + \sin\phi\,\hat{a}_y \\
&= \hat{a}_\rho
\end{aligned}
$$

A change in position from coordinates ρ, ϕ, z to $\rho + \Delta\rho, \phi, z$ results in a change in the position vector from $\overline{r}$ to $\overline{r}$ + $\overline{\Delta\ell}$. The vector $\overline{\Delta\ell}$ is a directed distance extending from point ρ, ϕ, z to point $\rho + \Delta\rho, \phi, z$, and is equal to:

$$
\begin{aligned}
\overline{\Delta\ell} &= \Delta\rho\,\frac{d\,\overline{r}}{d\rho} \\
&= \Delta\rho\ \cos\phi\ \hat{a}_x + \Delta\rho\ \sin\phi\ \hat{a}_y \\
&= \Delta\rho\,\hat{a}_\rho
\end{aligned}
$$



If $\Delta\ell$ is really small (i.e., as it approaches zero) we can define something called a **differential displacement vector** $\overline{d\ell}$:

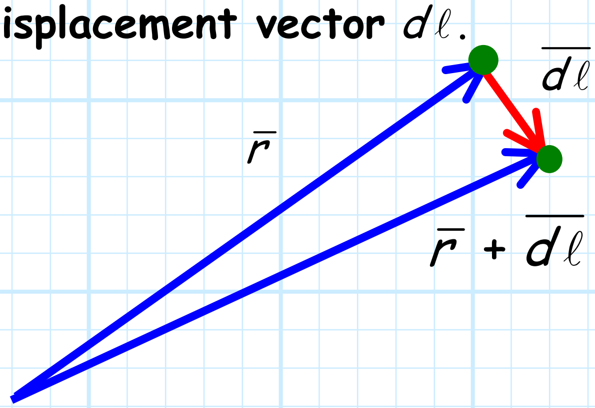
$$\overline{d\ell} \doteq \lim_{\Delta\ell \to 0} \overline{\Delta\ell}$$
$$= \lim_{\Delta\ell \to 0} \left(\frac{d\,\bar{r}}{d\,\ell} \right) \Delta\ell$$
$$= \left(\frac{d\,\bar{r}}{d\,\ell} \right) d\,\ell$$

For example:

$$\overline{d\rho} = \frac{d\,\bar{r}}{d\,\rho}\, d\,\rho = \hat{a}_\rho\, d\,\rho$$

Essentially, the differential line vector $\overline{d\ell}$ is the **tiny directed distance** formed when a point changes its location by some tiny amount, resulting in a change of one coordinate value ℓ by an equally tiny (i.e., differential) amount $d\ell$.

The **directed distance** between the original location (at coordinate value ℓ) and its new location (at coordinate value $\ell + d\ell$) is the **differential displacement vector** $\overline{d\ell}$.



We will use the differential line vector when evaluating a **line integral**.

The Differential Displacement Vector for Coordinate Systems

Let's determine the **differential displacement vectors** for each coordinate of the Cartesian, cylindrical and spherical coordinate systems!

Cartesian

This is easy!

$$\overline{dx} = \frac{d\bar{r}}{dx}\, dx = \left[\left(\frac{d\,x}{dx}\right)\hat{a}_x + \left(\frac{d\,y}{dx}\right)\hat{a}_y + \left(\frac{d\,z}{dx}\right)\hat{a}_z\right] dx$$

$$= \hat{a}_x\, dx$$

$$\overline{dy} = \frac{d\bar{r}}{dy}\, dy = \left[\left(\frac{d\,x}{dy}\right)\hat{a}_x + \left(\frac{d\,y}{dy}\right)\hat{a}_y + \left(\frac{d\,z}{dy}\right)\hat{a}_z\right] dy$$

$$= \hat{a}_y\, dy$$

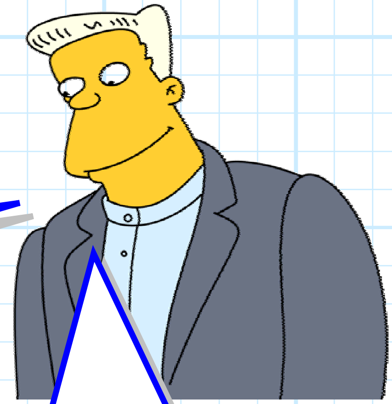
$$\overline{dz} = \frac{d\bar{r}}{dz}\, dz = \left[\left(\frac{d\,x}{dz}\right)\hat{a}_x + \left(\frac{d\,y}{dz}\right)\hat{a}_y + \left(\frac{d\,z}{dz}\right)\hat{a}_z\right] dz$$

$$= \hat{a}_z\, dz$$

Cylindrical

Likewise, recall from the last handout that:

$$\overline{d\rho} = \hat{a}_\rho \, d\rho$$



*Maria, look! I'm starting to see a **trend!***

$$\overline{dx} = \frac{d\bar{r}}{dx} \, dx = \hat{a}_x \, dx$$

$$\overline{dy} = \frac{d\bar{r}}{dy} \, dy = \hat{a}_y \, dy$$

$$\overline{dz} = \frac{d\bar{r}}{dz} \, dz = \hat{a}_z \, dz$$

$$\overline{d\rho} = \frac{d\bar{r}}{d\rho} \, d\rho = \hat{a}_\rho \, d\rho$$

Q: *It seems **very** apparent that:*

$$\overline{d\ell} = \hat{a}_\ell \, d\ell$$

*for **all** coordinates ℓ; right ?*

A: NO!! Do not make this mistake! For example, consider $\overline{d\phi}$:

Q: *No!! $\overline{d\phi} = \hat{a}_\phi \, \rho \, d\phi$?!?*

How did the coordinate ρ get in there?

$$\begin{aligned}
\overline{d\phi} &= \frac{d\bar{r}}{d\phi} d\phi \\
&= \left(\frac{dx}{d\phi} \hat{a}_x + \frac{dy}{d\phi} \hat{a}_y + \frac{dz}{d\phi} \hat{a}_z \right) d\phi \\
&= \left(\frac{d\,\rho\cos\phi}{d\phi} \hat{a}_x + \frac{d\,\rho\sin\phi}{d\phi} \hat{a}_y + \frac{dz}{d\phi} \hat{a}_z \right) d\phi \\
&= \left(-\rho\sin\phi\,\hat{a}_x + \rho\cos\phi\,\hat{a}_y \right) d\phi \\
&= \left(-\sin\phi\,\hat{a}_x + \cos\phi\,\hat{a}_y \right) \rho \, d\phi = \hat{a}_\phi \, \rho \, d\phi
\end{aligned}$$

The scalar differential value $\rho\, d\phi$ **makes sense**! The differential displacement vector is a **directed distance**, thus the units of its magnitude must be **distance** (e.g., meters, feet). The differential value $d\phi$ has units of **radians**, but the differential value $\rho\, d\phi$ **does** have units of distance.

The differential displacement vectors for the **cylindrical** coordinate system is therefore:

$$\overline{d\rho} = \frac{d\overline{r}}{d\rho} d\rho = \hat{a}_\rho \, d\rho$$

$$\overline{d\phi} = \frac{d\overline{r}}{d\phi} d\phi = \hat{a}_\phi \, \rho\, d\phi$$

$$\overline{dz} = \frac{d\overline{r}}{dz} dz = \hat{a}_z \, dz$$

Likewise, for the **spherical** coordinate system, we find that:

$$\overline{dr} = \frac{d\overline{r}}{dr} dr = \hat{a}_r \, dr$$

$$\overline{d\theta} = \frac{d\overline{r}}{d\theta} d\theta = \hat{a}_\theta \, r\, d\theta$$

$$\overline{d\phi} = \frac{d\overline{r}}{d\phi} d\phi = \hat{a}_\phi \, r \sin\theta\, d\phi$$

The Line Integral

This integral is alternatively known as the **contour integral**. The reason is that the line integral involves integrating the projection of a vector field onto a specified **contour** C, e.g.,

$$\int_C \mathbf{A}(\bar{r}_c) \cdot \overline{d\ell}$$

Some important things to note:

* The integrand is a **scalar** function.

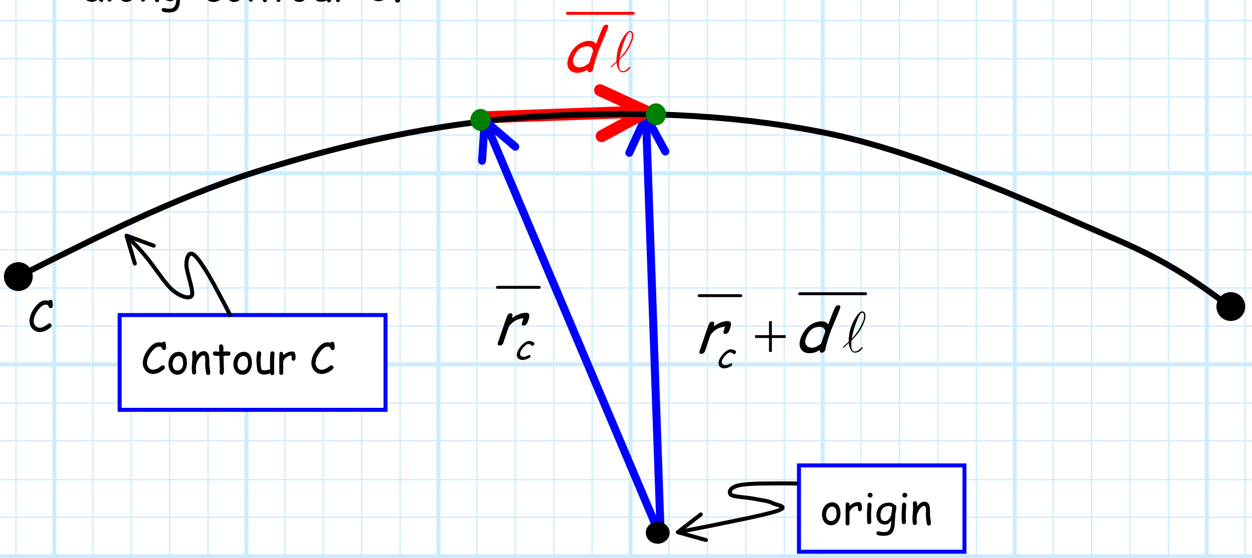
* The integration is over **one** dimension.

* The **contour** C is a line or curve through three-dimensional space.

* The position vector $\bar{r}_c$ denotes only those points that lie on contour C. Therefore, the value of this integral **only** depends on the value of vector field $\mathbf{A}(\bar{r})$ at the points along this contour.

Q: *What is the differential vector $\overline{d\ell}$, and how does it relate to contour C?*

A: The differential vector $\overline{d\ell}$ is the tiny **directed distance** formed when a point moves a small distance along contour C.



As a result, the differential line vector $\overline{d\ell}$ is **always tangential** to every point of the contour. In other words, the direction of $\overline{d\ell}$ always points "down" the contour.

Q: *So what does the scalar integrand $\mathbf{A}(\overline{r}_c)\cdot\overline{d\ell}$ mean? What is it that we are actually integrating?*

A: Essentially, the line integral integrates (i.e., "adds up") the values of a **scalar component** of vector field $\mathbf{A}(\overline{r})$ at **each and every point** along contour C. This scalar component of vector field $\mathbf{A}(\overline{r})$ is the projection of $\mathbf{A}(\overline{r}_c)$ onto the direction of the contour C.

First, I must point out that the notation $\mathbf{A}(\bar{r}_c)$ is **non-standard**. Typically, the vector field in the line integral is denoted simply as $\mathbf{A}(\bar{r})$. I use the notation $\mathbf{A}(\bar{r}_c)$ to emphasize that we are integrating the values of the vector field $\mathbf{A}(\bar{r})$ **only** at point that lie on contour C, and the points that lie on contour C are denoted as position vector $\bar{r}_c$.

In other words, the values of vector field $\mathbf{A}(\bar{r})$ at points that do not lie on the contour (which is just about all of them!) have no effect on the integration. The integral **only** depends on the value of the vector field as we move along contour C—we denote these values as $\mathbf{A}(\bar{r}_c)$.

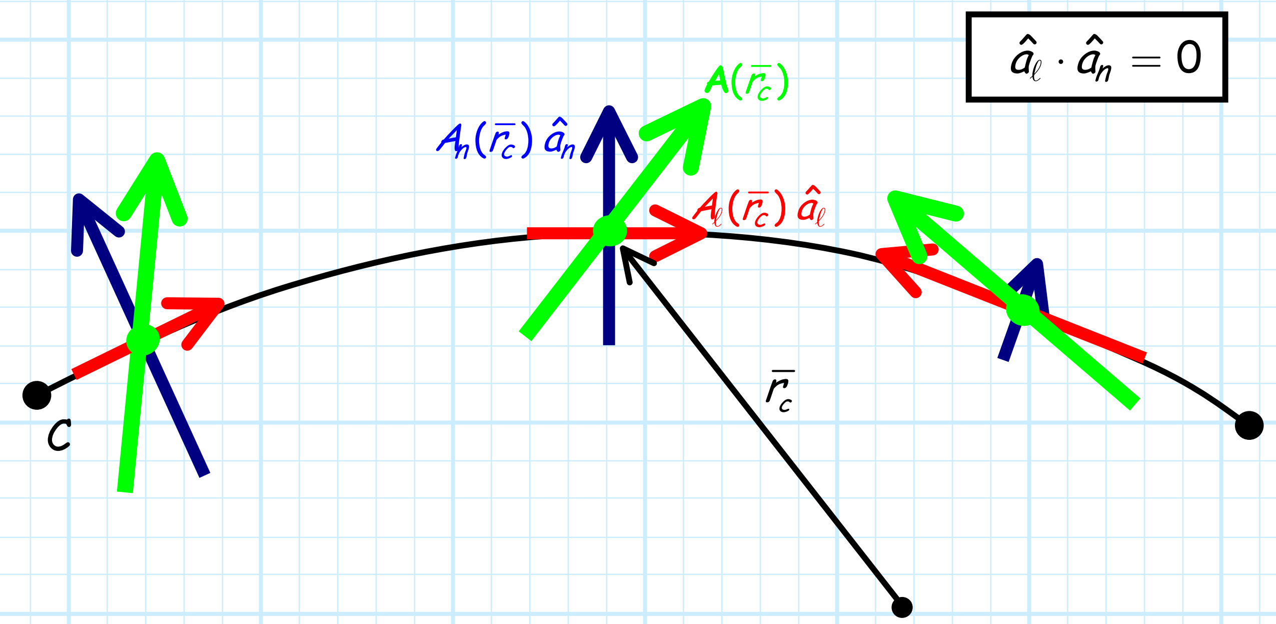
Moreover, the line integral depends on **only one scalar component** of $\mathbf{A}(\bar{r}_c)$!

Q: *On just what component of $\mathbf{A}(\bar{r}_c)$* ***does*** *the integral depend?*

A: Look at the integrand $\mathbf{A}(\bar{r}_c)\cdot\overline{d\ell}$ —we see it involves the **dot** product! Thus, we find that the scalar integrand is simply the **scalar projection** of $\mathbf{A}(\bar{r}_c)$ onto the differential vector $\overline{d\ell}$. As a result, the integrand depends **only** the component of $\mathbf{A}(\bar{r}_c)$ that lies in the direction of $\overline{d\ell}$—and $\overline{d\ell}$ **always** points in the direction of the contour C!

To help see this, first note that $\mathbf{A}(\bar{r}_c)$, the value of the vector field along the contour, can be written in terms of a vector component **tangential** to the contour (i.e, $A_\ell(\bar{r}_c)\,\hat{a}_\ell$), and a vector component that is **normal** (i.e., orthogonal) to the contour (i.e, $A_n(\bar{r}_c)\,\hat{a}_n$):

$$\mathbf{A}(\bar{r}_c) = A_\ell(\bar{r}_c)\,\hat{a}_\ell + A_n(\bar{r}_c)\,\hat{a}_n$$



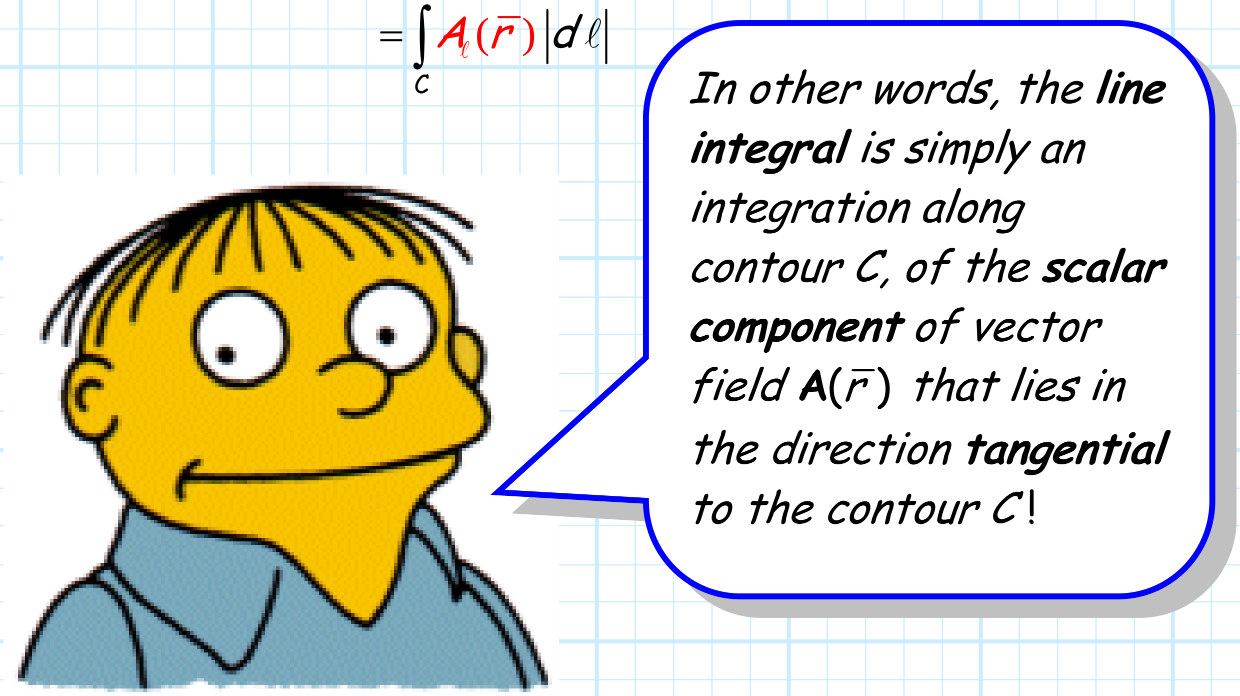
We likewise note that the differential line vector $\overline{d\ell}$, like **any** and all vectors, can be written in terms of its magnitude ($|d\ell|$) and direction ($\hat{a}_\ell$) as:

$$\overline{d\ell} = \hat{a}_\ell\,|d\ell|$$

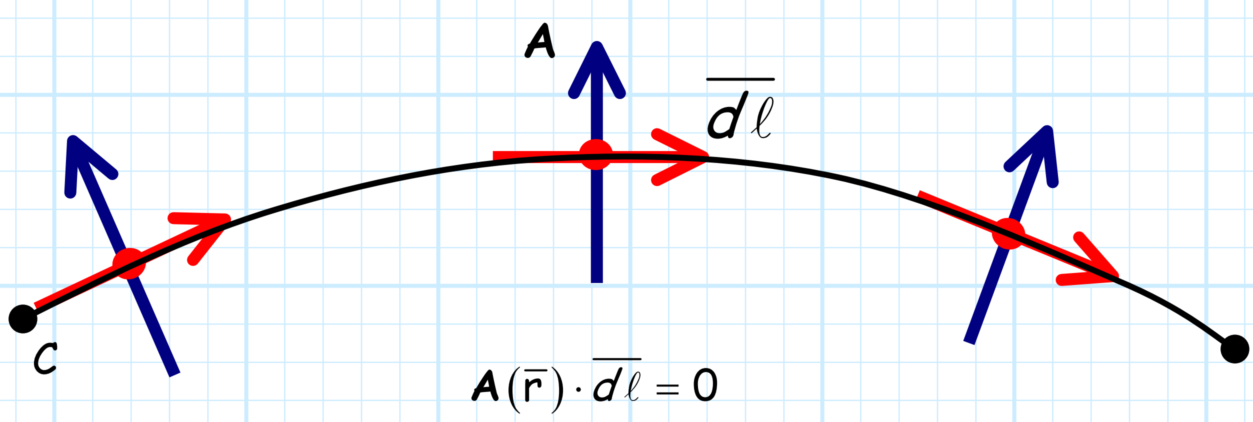
For example, for $\overline{d\phi} = \rho\, d\phi\, \hat{a}_\phi$, we can say $|d\ell| = \rho\, d\phi$ and $\hat{a}_\ell = \hat{a}_\phi$.

As a result we can write:

$$\begin{aligned}\int_C \mathbf{A}(\bar{r}_c)\cdot\overline{d\ell} &= \int_C \left[A_\ell(\bar{r})\,\hat{a}_\ell + A_n(\bar{r})\,\hat{a}_n\right]\cdot\overline{d\ell}\\ &= \int_C \left[A_\ell(\bar{r})\,\hat{a}_\ell + A_n(\bar{r})\,\hat{a}_n\right]\cdot\ \hat{a}_\ell\,|d\ell|\\ &= \int_C \left[A_\ell(\bar{r})\,\hat{a}_\ell\cdot\hat{a}_\ell + A_n(\bar{r})\,\hat{a}_n\cdot\hat{a}_\ell\right]|d\ell|\\ &= \int_C A_\ell(\bar{r})\,|d\ell|\end{aligned}$$



Note if vector field $\mathbf{A}(\bar{r})$ is **orthogonal** to the contour at every point, then the resulting line integral will be **zero**.



Although C represents **any** contour, no matter how **complex** or **convoluted**, we will study only **basic** contours. In other words, $\overline{d\ell}$ will correspond to one of the differential line vectors we have **previously** determined for Cartesian, cylindrical, and spherical coordinate systems.

The Contour C

In this class, we will limit ourselves to studying only those contours that are formed when we change the location of a point by varying **just one** coordinate parameter. In other words, the other two coordinate parameters will remain **fixed**.

Mathematically, therefore, a **contour** is described by:

2 equalities (e.g., $x=2$, $y=-4$; $r=3$, $\phi=\pi/4$)

AND

1 inequality (e.g., $-1 < z < 5$; $0 < \theta < \pi/2$)

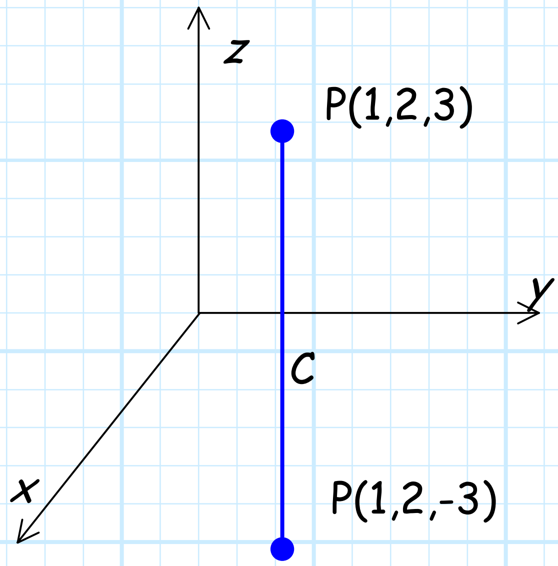
Likewise, we will need to explicitly determine the differential displacement vector $\overline{d\ell}$ for each contour.

Recall we have studied **seven** coordinate parameters ($x, y, z, \rho, \phi, r, \theta$). As a result, we can form **seven** different contours C!

Cartesian Contours

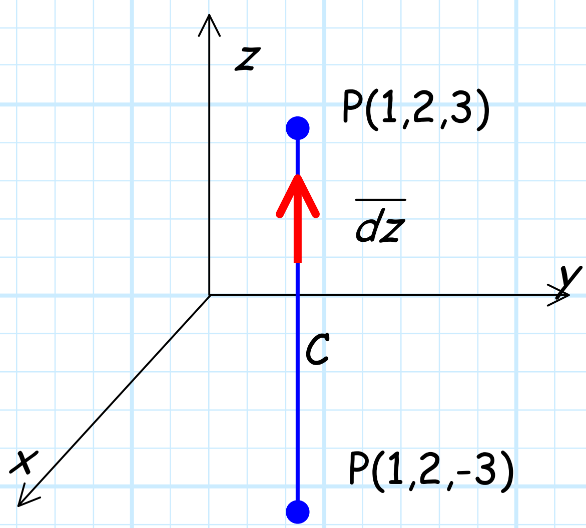
Say we move a point from P($x=1$, $y=2$, $z=-3$) to P($x=1$, $y=2$, $z=3$) by changing **only** the coordinate variable z from $z=-3$ to $z=3$. In other words, the coordinate values x and y remain **constant** at $x=1$ and $y=2$.

We form a contour that is a **line segment**, **parallel** to the *z*-axis!



Note that **every** point along this segment has coordinate values $x=1$ and $y=2$. As we move along the contour, the **only** coordinate value that changes is *z*.

Therefore, the **differential** directed distance associated with a change in position from *z* to *z+dz*, is $\overline{d\ell}=\overline{dz}=\hat{a}_z dz$.



Similarly, a line segment parallel to the x-axis (or y-axis) can be formed by changing coordinate parameter x (or y), with a resulting differential displacement vector of $\overline{d\ell} = \overline{dx} = \hat{a}_x dx$ (or $\overline{d\ell} = \overline{dy} = \hat{a}_y dy$).

The three Cartesian contours are therefore:

1. Line segment parallel to the z-axis

$$x = c_x \qquad y = c_y \qquad c_{z1} \leq z \leq c_{z2}$$

$$\overline{d\ell} = \hat{a}_z \, dz$$

2. Line segment parallel to the y-axis

$$x = c_x \qquad c_{y1} \leq y \leq c_{y2} \qquad z = c_z$$

$$\overline{d\ell} = \hat{a}_y \, dy$$

3. Line segment parallel to the x-axis

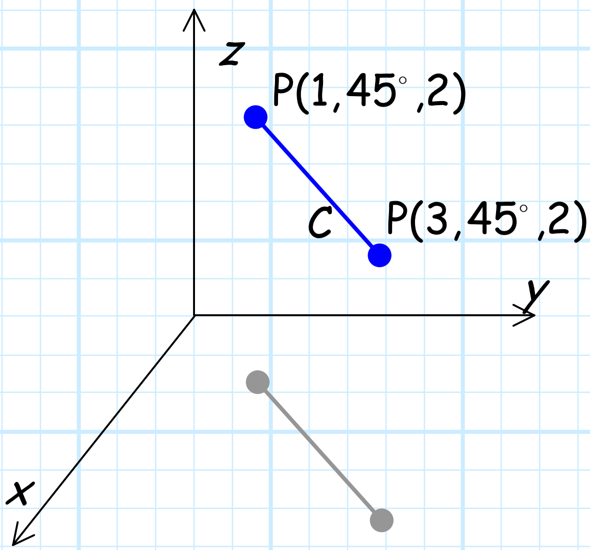
$$c_{x1} \leq x \leq c_{x2} \qquad y = c_y \qquad z = c_z$$

$$\overline{d\ell} = \hat{a}_x \, dx$$

Cylindrical Contours

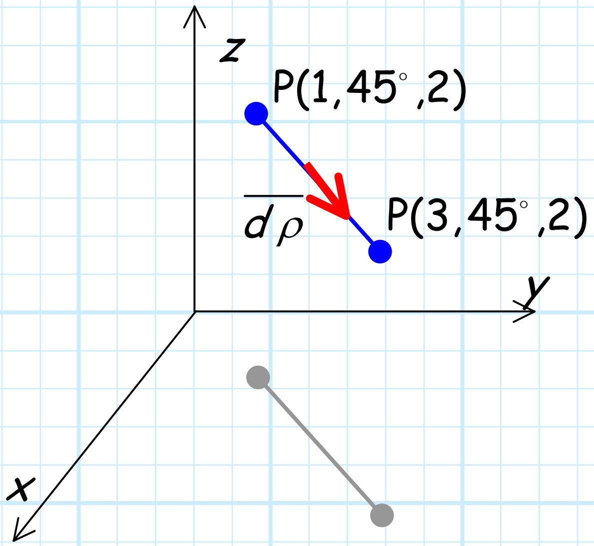
Say we move a point from P(ρ=1, $\phi = 45^\circ$, z =2) to P(ρ=3, $\phi = 45^\circ$, z =2) by changing **only** the coordinate variable ρ from ρ =*1* to ρ=3. In other words, the coordinate values ϕ and z remain **constant** at $\phi = 45^\circ$ and z=2.

We form a contour that is a **line segment**, **parallel** to the *x-y* plane (i.e., perpendicular to the *z*-axis).



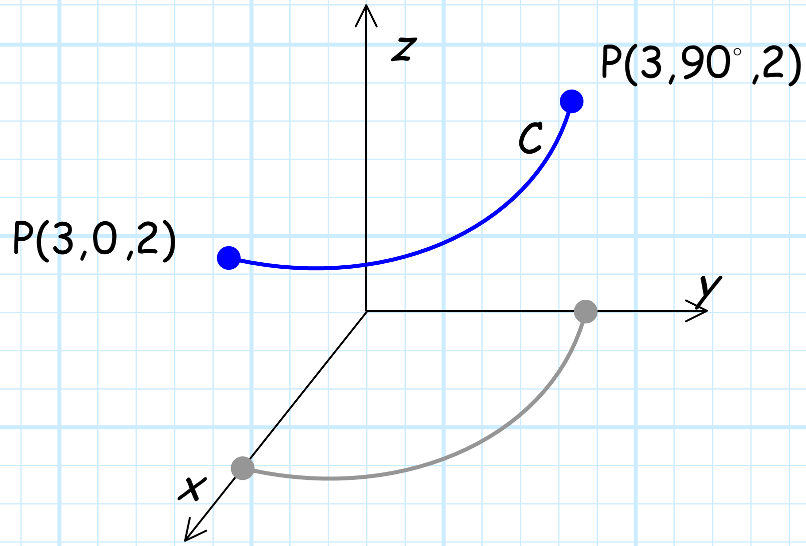
Note that **every** point along this segment has coordinate values $\phi = 45^\circ$ and z=2. As we move along the contour, the **only** coordinate value that changes is ρ.

Therefore, the **differential** directed distance associated with a change in position from ρ to $\rho + d\rho$, is $\overline{d\ell} = \overline{d\rho} = \hat{a}_\rho \, d\rho$.



Alternatively, say we move a point from P(ρ=3, $\phi = 0$, z=2) to P(ρ=3, $\phi = 90^\circ$, z=2) by changing **only** the coordinate variable ϕ from ϕ=0 to $\phi = 90^\circ$. In other words, the coordinate values ρ and z remain **constant** at ρ=3 and z=2.

We form a contour that is a **circular arc**, parallel to the x-y plane.

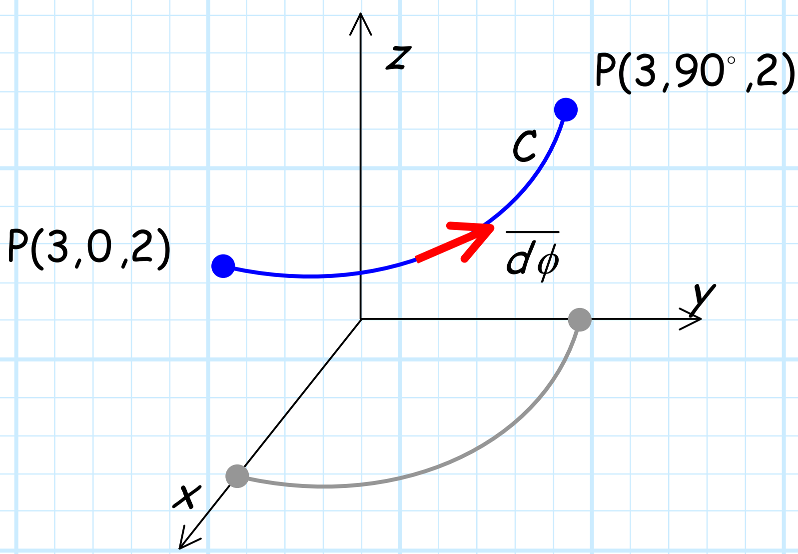


Note: if we move from $\phi = 0$ to $\phi = 360^\circ$, a complete **circle** is formed around the z-axis.

Every point along the arc has coordinate values $\rho = 3$ and z =2. As we move along the contour, the **only** coordinate value that changes is ϕ.

Therefore, the **differential** directed distance associated with a change in position from ϕ to $\phi + d\phi$, is:

$$\overline{d\ell} = \overline{d\phi} = \hat{a}_\phi \ \rho d\phi \,.$$



Finally, changing coordinate z generates the **third** cylindrical contour—but we **already** did that in Cartesian coordinates! The result is **again** a line segment parallel to the z-axis.

The three cylindrical contours are therefore described as:

1. *Line segment parallel to the z- axis.*

$$\rho = c_\rho \qquad \phi = c_\phi \qquad c_{z1} \leq z \leq c_{z2}$$

$$\overline{d\ell} = \hat{a}_z \, dz$$

2. *Circular arc parallel to the x-y plane.*

$$\rho = c_\rho \qquad c_{\phi 1} \leq \phi \leq c_{\phi 2} \qquad z = c_z$$

$$\overline{d\ell} = \hat{a}_\phi \;\; \rho d\phi$$

3. *Line segment parallel to the x-y plane.*

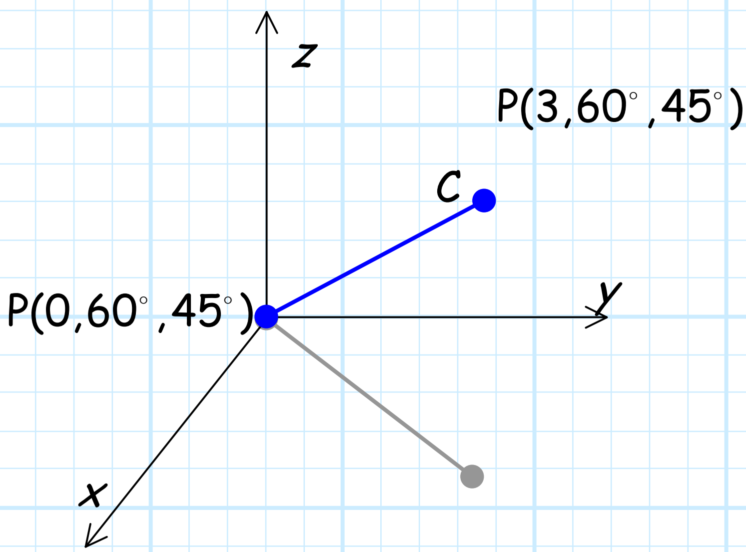
$$c_{\rho 1} \leq \rho \leq c_{\rho 2} \qquad \phi = c_\phi \qquad z = c_z$$

$$\overline{d\ell} = \hat{a}_\rho \;\; d\rho$$

Spherical Contours

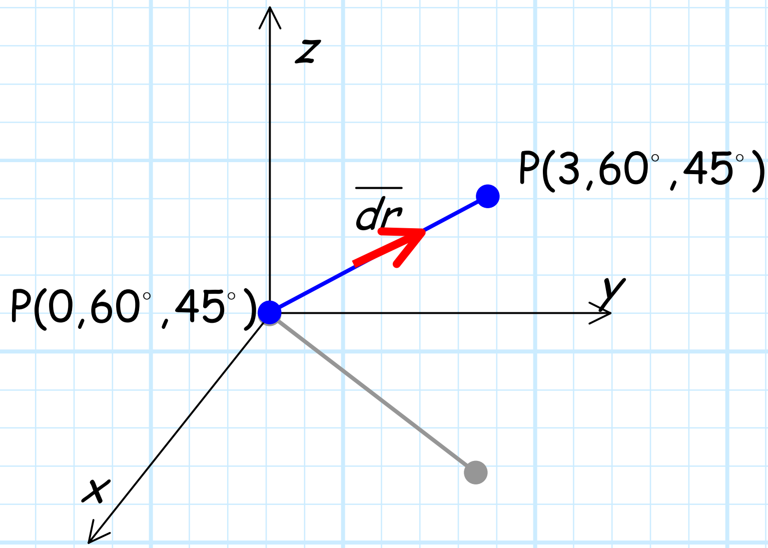
Say we move a point from P(r =0, $\theta = 60^\circ$, $\phi = 45^\circ$) to P(r =3, $\theta = 60^\circ$, $\phi = 45^\circ$) by changing **only** the coordinate variable r from r=0 to r =3. In other words, the coordinate values θ and ϕ remain **constant** at $\theta = 60^\circ$ and $\phi = 45^\circ$.

We form a contour that is a **line segment**, emerging from the **origin**.



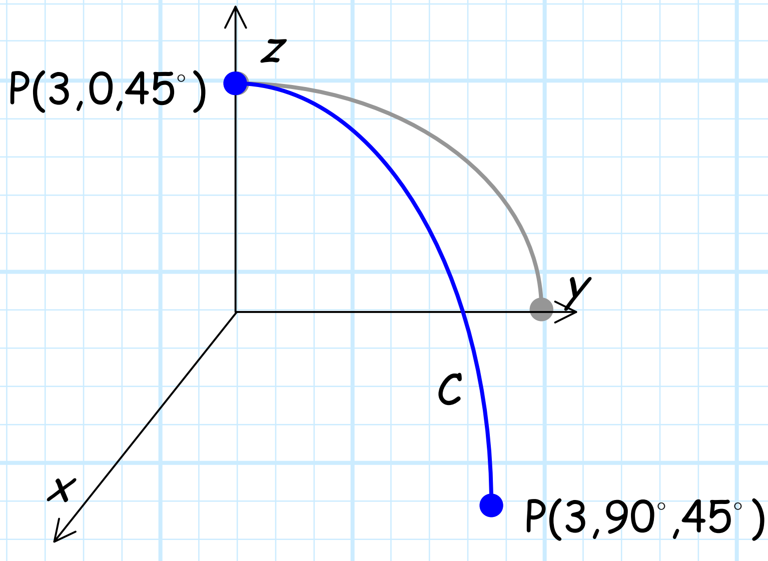
Every point along the line segment has coordinate values $\theta = 60^\circ$ and $\phi = 45^\circ$. As we move along the contour, the **only** coordinate value that changes is r.

Therefore, the **differential** directed distance associated with a change in position from r to $r+dr$, is $\overline{d\ell} = \overline{dr} = \hat{a}_r\, dr$.



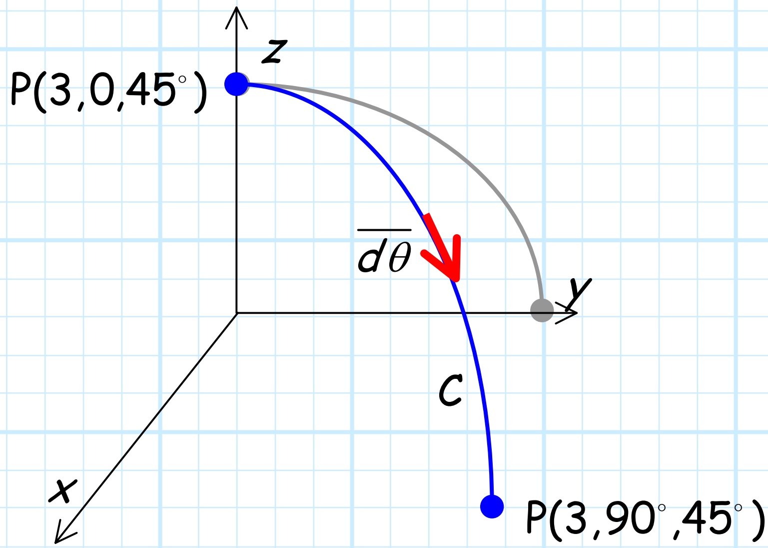
Alternatively, say we move a point from P(r=3, $\theta = 0$, $\phi = 45^\circ$) to P(r=3, $\theta = 90^\circ$, $\phi = 45^\circ$) by changing **only** the coordinate variable θ from $\theta = 0$ to $\theta = 90^\circ$. In other words, the coordinate values θ and ϕ remain **constant** at $\theta = 60^\circ$ and $\phi = 45^\circ$.

We form a **circular arc**, whose plane includes the z-axis.



Every point along the arc has coordinate values $r = 3$ and $\phi = 45^\circ$. As we move along the contour, the **only** coordinate value that changes is θ.

Therefore, the **differential** directed distance associated with a change in position from θ to $\theta + d\theta$, is $\overline{d\ell} = \overline{d\theta} = \hat{a}_\theta \, r \, d\theta$.



Finally, we could fix coordinates r and θ and vary coordinate ϕ only—but we **already** did this in cylindrical coordinates! We **again** find that a **circular arc** is generated, an arc that is parallel to the x-y plane.

The three spherical contours are therefore:

1. *A circular arc parallel to the x-y plane.*

$$r = c_r \qquad \theta = c_\theta \qquad c_{\phi 1} \leq \phi \leq c_{\phi 2}$$

$$\overline{d\ell} = \hat{a}_\phi \;\; r \sin\theta \, d\phi$$

2. *A circular arc in a plane that includes the z-axis.*

$$r = c_r \qquad c_{\theta 1} \leq \theta \leq c_{\theta 2} \qquad \phi = c_\phi$$

$$\overline{d\ell} = \hat{a}_\theta \;\; r \, d\theta$$

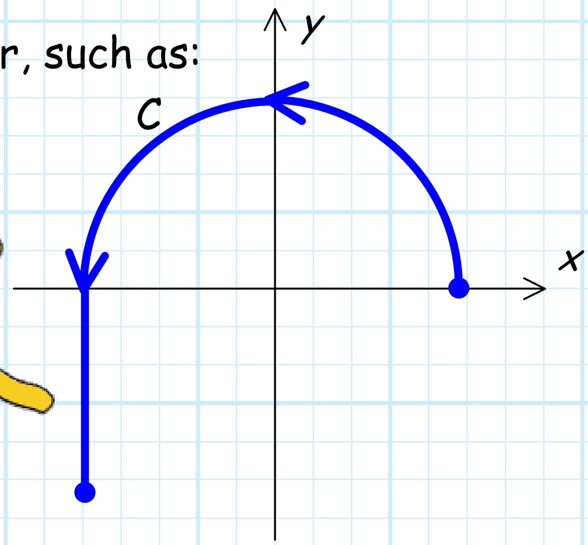
3. *A line segment directed toward the origin.*

$$c_{r1} \leq r \leq c_{r2} \qquad \theta = c_\theta \qquad \phi = c_\phi$$

$$\overline{d\ell} = \hat{a}_r \;\; dr$$

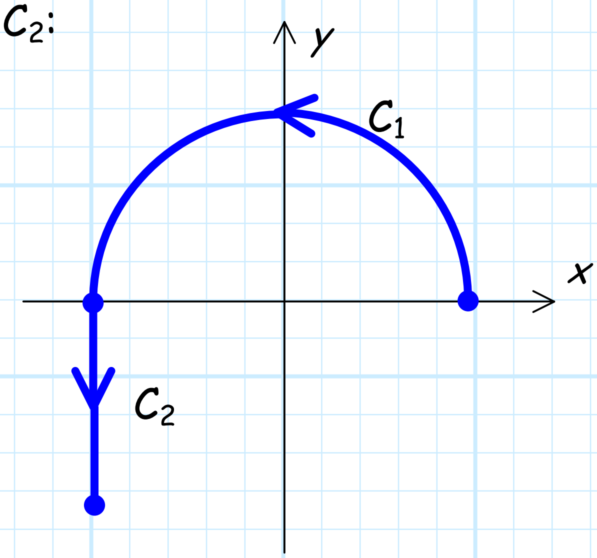
Line Integrals with Complex Contours

Consider a more **complex** contour, such as:



Q: *What's this flim-flam?! **This** contour can **neither** be expressed in terms of **single** coordinate inequality, **nor** with **single** differential line vector!*

A: True! But we can still **easily** evaluate a line integral over this contour C. The trick is to divide C into **two** contours, denoted as C_1 and C_2:



We can denote contour C as $C = C_1 + C_2$. It can be shown that:

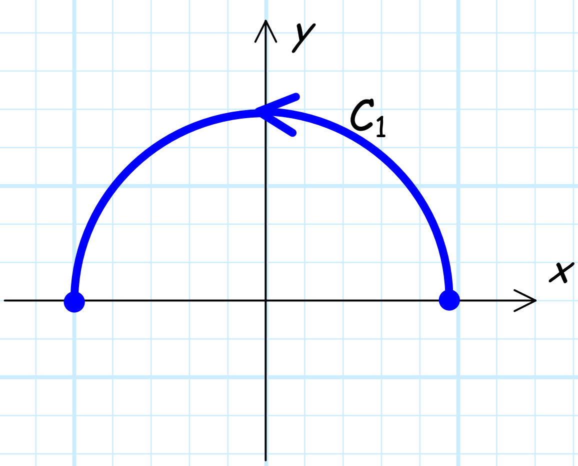
$$\int_C \mathbf{A}(\bar{r}_c)\cdot\overline{d\ell} = \int_{C_1} \mathbf{A}(\bar{r}_c)\cdot\overline{d\ell} + \int_{C_2} \mathbf{A}(\bar{r}_c)\cdot\overline{d\ell}$$

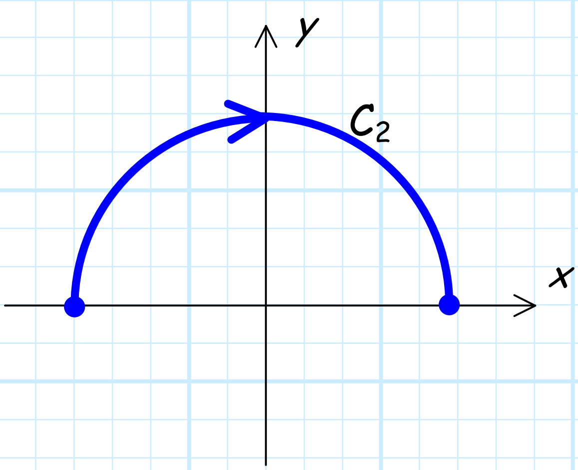
Note for the example given, we can evaluate the integral over both contour C_1 and contour C_2. The first is a **circular arc** around the z-axis, and the second is a **line segment** parallel to the y-axis.

Q: *Does the direction of the contour matter?*

A: **YES!** Every contour has a **starting** point and an **end** point. Integrating along the contour in the **opposite** direction will result in an **incorrect** answer!

For example, consider the two contours below:





In this case, the two contours are identical, with the **exception** of **direction**. In other words the beginning point of one is the end point of the other, and vice versa.

For this example, we would relate the two contours by saying:

$$C_1 = -C_2 \quad \text{and/or} \quad C_2 = -C_1$$

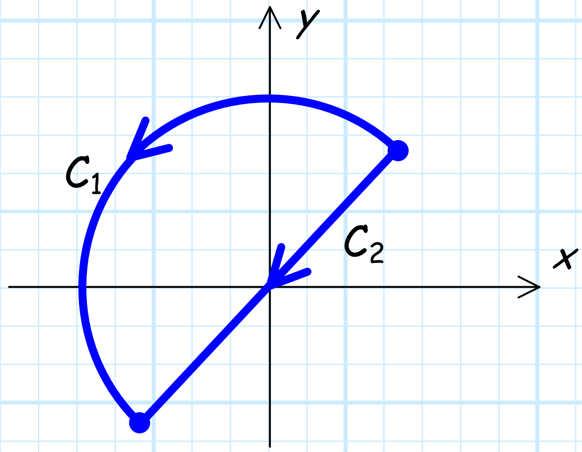
Just like vectors, the **negative** of a contour is an otherwise identical contour with opposite direction. We find that:

$$\int_{-C} \mathbf{A}(\bar{r}_c) \cdot \overline{d\ell} = -\int_{C} \mathbf{A}(\bar{r}_c) \cdot \overline{d\ell}$$

Q: *Does the **shape** of the contour **really** matter, or does the result of line integration only depend on the starting and end points ??*

A: **Generally speaking**, the shape of the contour **does** matter. Not only does the line integral depend on where we start and where we finish, it **also** depends on the path we take to get there!

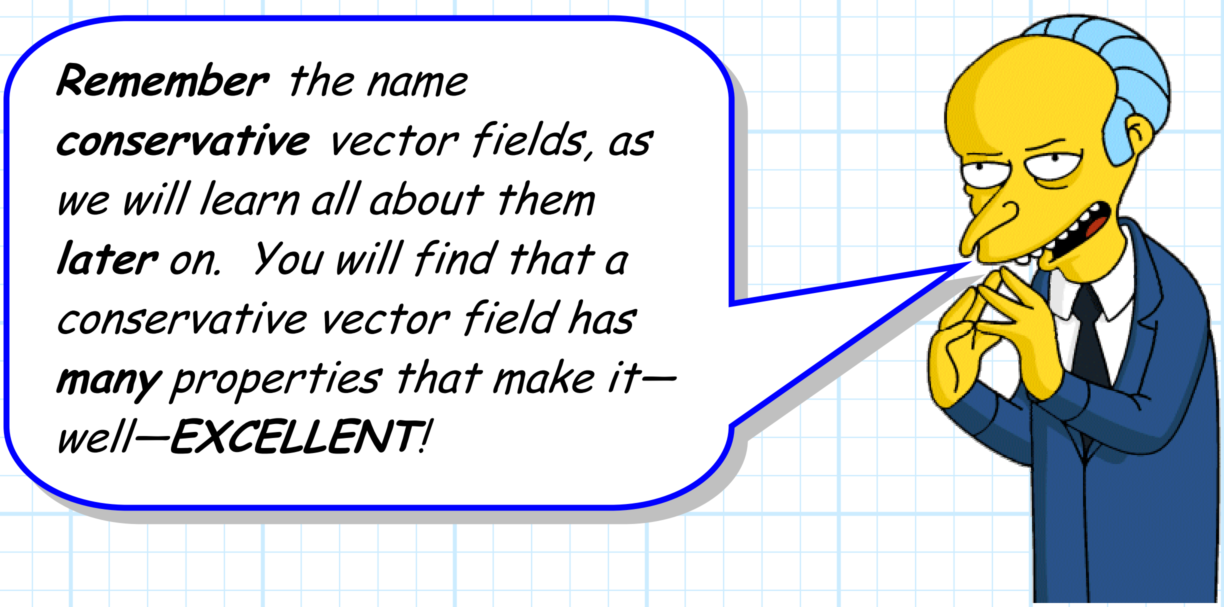
For example, consider these two contours:



Generally speaking, we find that:

$$\int_{C_1} \mathbf{A}(\bar{r}_c)\cdot\overline{d\ell} \neq \int_{C_2} \mathbf{A}(\bar{r}_c)\cdot\overline{d\ell}$$

An **exception** to this is a **special** category of vector fields called **conservative** fields. For conservative fields, the contour path does **not** matter—the beginning and end points of the contour are **all** that are required to evaluate a line integral !



Steps for Analyzing Line Integrals

You wish to evaluate an integral of the form:

$$\int_C \mathbf{A}(\bar{r}_c) \cdot \overline{d\ell}$$

To successfully accomplish this, simply follow **these** steps:

Step 1: Determine the 2 equalities, 1 inequality, and $\overline{d\ell}$ for the **contour** C.

Step 2: Evaluate the **dot product** $\mathbf{A}(\bar{r}) \cdot \overline{d\ell}$.

Step 3: Transform all coordinates of the resulting **scalar** field to the **same** system as C.

Step 4: Evaluate the scalar field using the **two** coordinate **equalities** that describe contour C.

Step 5: Determine the **limits of integration** from the **inequality** that describes contour C *(be careful of order!)*.

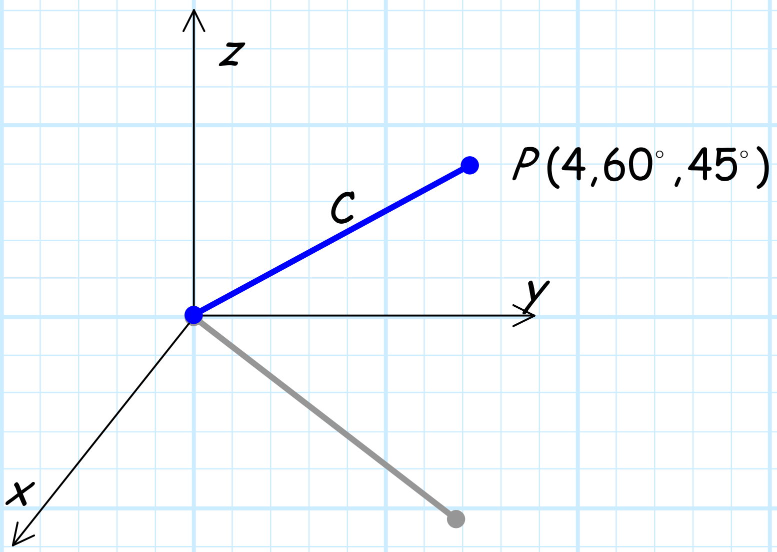
Step 6: Integrate the remaining function of **one** coordinate variable.

Example: The Line Integral

Consider the vector field:

$$\mathbf{A}(\bar{r}_c) = z\ \hat{a}_x - x\ \hat{a}_y$$

Integrate this vector field over **contour** C, a straight line that begins at the **origin** and ends at point $P(r = 4, \theta = 60^\circ, \phi = 45^\circ)$.



Step 1: Determine the **two equalities**, **one inequality**, and proper $\overline{d\ell}$ for the **contour** C.

This contour is formed as the coordinate r changes from r=0 to r=4, where $\theta = 60^\circ$ and $\phi = 45^\circ$ for all points. The two equalities and one inequality that define this contour are thus:

$$0 \leq r \leq 4 \qquad \theta = 60^\circ \qquad \phi = 45^\circ$$

and the **differential** displacement vector for this contour is therefore:

$$\overline{d\ell} = \overline{dr} = \hat{\mathbf{a}}_r \, dr$$

Step 2: Evaluate the **dot product** $\mathbf{A}(\bar{r}_c) \cdot \overline{d\ell}$.

$$\begin{aligned}\mathbf{A}(\bar{r}_c) \cdot \overline{d\ell} &= \left(z \ \hat{a}_x - x \ \hat{a}_y\right) \cdot \hat{a}_r \, dr \\ &= \left(z \ \hat{a}_x \cdot \hat{a}_r - x \ \hat{a}_y \cdot \hat{a}_r\right) dr \\ &= (z \ \sin\theta\cos\phi - x \ \sin\theta\sin\phi) \, dr\end{aligned}$$

Step 3: Transform all coordinates of the resulting **scalar** field to the **same** system as C.

The contour is a **spherical** contour. Recall that $z = r\cos\theta$ and $x = r\sin\theta\cos\phi$, therefore:

$$\begin{aligned}\mathbf{A}(\bar{r}_c) \cdot \overline{d\ell} &= (z \ \sin\theta\cos\phi - x \ \sin\theta\sin\phi) \, dr \\ &= (r\cos\theta \ \sin\theta\cos\phi - r\sin\theta\cos\phi \ \sin\theta\sin\phi) \, dr \\ &= r \ \sin\theta\cos\phi(\cos\theta - \sin\theta\sin\phi) dr\end{aligned}$$

Step 4: Evaluate the scalar field using the **two** coordinate **equalities** that describe contour C.

Recall that θ=60° and ϕ=45° at **every** point along the contour we are integrating over. Thus, functions of θ or φ are **constants** with respect to the integration! For example, $\cos\theta = \cos 45^\circ = 0.5$. Therefore:

$$\mathbf{A}(\overline{r}_c)\cdot\overline{d\ell} = r\ \sin 60^\circ \cos 45^\circ \left(\cos 60^\circ - \sin 60^\circ \sin 45^\circ\right) dr$$
$$= r\ \sqrt{3/4}\ \sqrt{1/2}\left(1/2 - \sqrt{3/4}\sqrt{1/2}\right) dr$$
$$= r\sqrt{3/8}\left(\frac{\sqrt{2}-\sqrt{3}}{\sqrt{8}}\right) dr$$
$$= \left(\frac{\sqrt{6}-3}{8}\right) r\ dr$$

Step 5: Determine the **limits of integration** from the **inequality** that describes contour *C (be careful of order!)*.

We note the contour is described as:

$$0 \leq r \leq 4$$

and the contour *C* moves from $r = 0$ to $r = 4$. Thus, we integrate from 0 to 4 :

$$\int_C \mathbf{A}(\overline{r}_c)\cdot\overline{d\ell} = \int_0^4 \left(\frac{\sqrt{6}-3}{8}\right) r\ dr$$

Note: if the contour ran from $r = 4$ to $r = 0$ the limits of integration would be **flipped**! I.E.,

$$\int_4^0 \left(\frac{\sqrt{6}-3}{8}\right) r\ dr$$

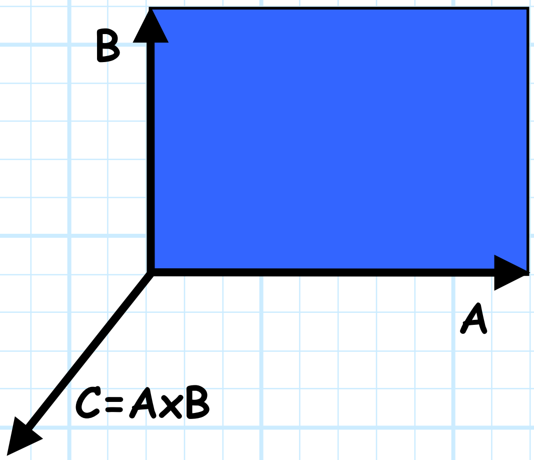
It is readily apparent that the line integral from $r = 0$ to $r = 4$ is the opposite (i.e., **negative**) of the integral from $r = 4$ to $r = 0$.

Step 6: Integrate the remaining function of **one** coordinate variable.

$$\begin{aligned}
\int_C \mathbf{A}(\bar{r}_c) \cdot \overline{d\ell} &= \int_0^4 \left(\frac{\sqrt{6}-3}{8}\right) r\, dr \\
&= \left(\frac{\sqrt{6}-3}{8}\right) \int_0^4 r\, dr \\
&= \left(\frac{\sqrt{6}-3}{8}\right)\left(\frac{4^2}{2} - \frac{0^2}{2}\right) \\
&= \sqrt{6} - 3
\end{aligned}$$

Differential Surface Vectors

Consider a **rectangular surface**, oriented in some arbitrary direction:



We can describe this surface using **vectors**! One vector (say **A**), is a directed distance that denotes the **length** (i.e, magnitude) and **orientation** of one edge of the rectangle, while another directed distance (say **B**) denotes the length and orientation of the other edge.

Say we take the **cross-product** of these two vectors (**AxB=C**).

Q: *What does this vector **C** represent?*

A: Look at the **definition** of cross product!

$$\begin{aligned} \mathbf{C} &= \mathbf{A} \times \mathbf{B} \\ &= \hat{a}_n \left|\mathbf{A}\right| \left|\mathbf{B}\right| \sin\theta_{AB} \\ &= \hat{a}_n \left|\mathbf{A}\right| \left|\mathbf{B}\right| \end{aligned}$$

Note that:

$$\left|\mathbf{C}\right| = \left|\mathbf{A}\right| \left|\mathbf{B}\right|$$

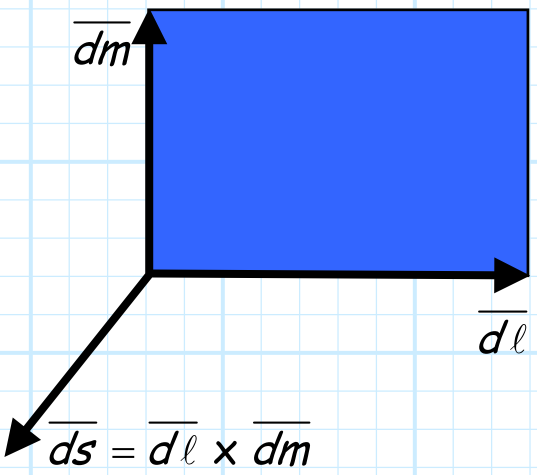
The magnitude of vector **C** is therefore product of the lengths of each directed distance—the **area of the rectangle**!

Likewise, $\mathbf{C} \cdot \mathbf{A} = 0$ and $\mathbf{C} \cdot \mathbf{B} = 0$, therefore vector **C** is orthogonal (i.e., "**normal**") to the **surface** of the rectangle.

As a result, vector **C** indicates **both** the **size** and the **orientation** of the rectangle.

The differential surface vector

For example, consider the **very small** rectangular surface resulting from two differential displacement vectors, say $\overline{d\ell}$ and $\overline{dm}$.



For example, consider the situation if $\overline{d\ell} = \overline{dx}$ and $\overline{dm} = \overline{dy}$:

$$\begin{aligned} \overline{ds} &= \overline{dx} \times \overline{dy} \\ &= \left(\hat{a}_x \times \hat{a}_y \right) dx\, dy \\ &= \hat{a}_z\, dx\, dy \end{aligned}$$

In other words the **differential** surface element has an **area** equal to the product $dx\, dy$, and a **normal vector** that points in the $\hat{a}_z$ direction.

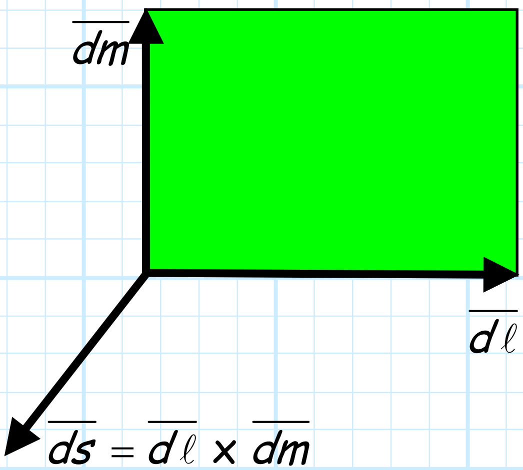
The differential surface vector $\overline{ds}$ specifies the size and orientation of a small (i.e., **differential**) **patch** of area, located on some arbitrary **surface** S.

We will use the differential surface vector in evaluating **surface integrals** of the type:

$$\iint_S \mathbf{A}(\overline{r}_s) \cdot \overline{ds}$$

The Differential Surface Vector for Coordinate Systems

Given that $\overline{ds} = \overline{d\ell} \times \overline{dm}$, we can determine the differential surface vectors for each of the **three** coordinate systems.



Cartesian

$$\overline{ds_x} = \overline{dy} \times \overline{dz} = \hat{a}_x \, dy\, dz$$
$$\overline{ds_y} = \overline{dz} \times \overline{dx} = \hat{a}_y \, dx\, dz$$
$$\overline{ds_z} = \overline{dx} \times \overline{dy} = \hat{a}_z \, dx\, dy$$

We shall find that these differential surface vectors define a small patch of area on the surface of **flat plane**.

Cylindrical

$$\overline{ds_\rho} = \overline{d\phi} \times \overline{dz} = \hat{a}_\rho \;\rho\, d\phi\, dz$$
$$\overline{ds_\phi} = \overline{dz} \times \overline{dp} = \hat{a}_\phi \; d\rho\, dz$$
$$\overline{ds_z} = \overline{d\rho} \times \overline{d\phi} = \hat{a}_z \;\rho\, d\rho\, d\phi$$

We shall find that $\overline{ds_\rho}$ describes a small patch of area on the surface of a **cylinder**, $\overline{ds_\phi}$ describes a small patch of area on the surface of a **half-plane**, and $\overline{ds_z}$ again describes a small patch of area on the surface of a flat **plane**.

Spherical

$$\overline{ds_r} = \overline{d\theta} \times \overline{d\phi} = \hat{a}_r \; r^2 \sin\theta \; d\theta\, d\phi$$
$$\overline{ds_\theta} = \overline{d\phi} \times \overline{dr} = \hat{a}_\theta \; r \sin\theta \; dr\, d\phi$$
$$\overline{ds_\phi} = \overline{dr} \times \overline{d\theta} = \hat{a}_\phi \; r \; dr\, d\theta$$

We shall find that $\overline{ds_r}$ describes a small patch of area on the surface of a **sphere**, $\overline{ds_\theta}$ describes a small patch of area on the surface of a **cone**, and $\overline{ds_\phi}$ again describes a small patch of area on the surface of a **half plane**.

The Surface Integral

An important type of vector integral that is often quite useful for solving physical problems is the **surface integral**:

$$\iint_S \mathbf{A}(\bar{r}_s) \cdot \overline{ds}$$

Some important things to note:

* The integrand is a **scalar** function.

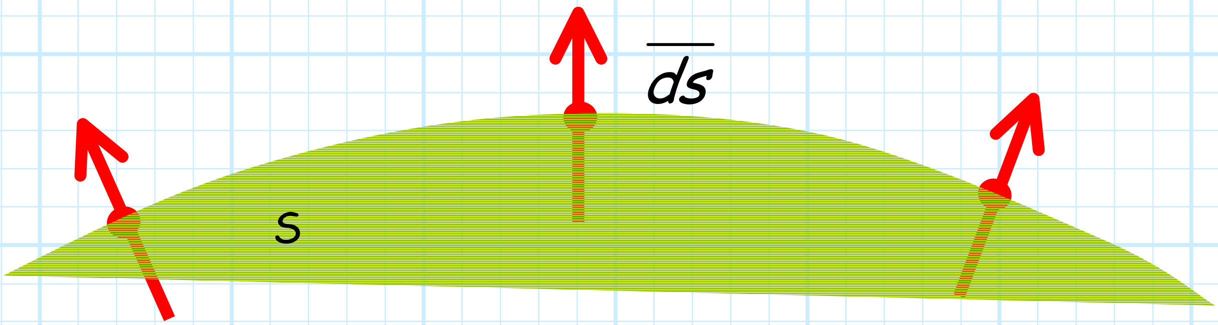
* The integration is over **two** dimensions.

* The **surface** S is an arbitrary two-dimensional surface in a three-dimensional space.

* The position vector $\bar{r}_s$ denotes only those points that lie on surface S. Therefore, the value of this integral **only** depends on the value of vector field $\mathbf{A}(\bar{r})$ at the points on this surface.

Q: *How are differential surface vector $\overline{ds}$ and surface S related?*

A: The differential vector $\overline{ds}$ describes a differential surface area at every point on S.



As a result, the differential surface vector $\overline{ds}$ is **normal** (i.e., orthogonal) to surface S at every point on S.

Q: *So what does the scalar integrand $\mathbf{A}(\bar{r}_s)\cdot\overline{ds}$ mean? What is it that we are actually integrating?*

A: Essentially, the surface integral integrates (i.e., "adds up") the values of a **scalar component** of vector field $\mathbf{A}(\bar{r})$ at **each and every point** on surface S. This scalar component of vector field $\mathbf{A}(\bar{r})$ is the projection of $\mathbf{A}(\bar{r}_s)$ onto a direction perpendicular (i.e., normal) to the surface S.

First, I must point out that the notation $\mathbf{A}(\bar{r}_s)$ is **non-standard**. Typically, the vector field in the surface integral is denoted simply as $\mathbf{A}(\bar{r})$. I use the notation $\mathbf{A}(\bar{r}_s)$ to emphasize that we are integrating the values of the vector field $\mathbf{A}(\bar{r})$ **only** at points that lie on surface S, and the points that lie on surface S are denoted by position vector $\bar{r}_s$.

In other words, the values of vector field $\mathbf{A}(\bar{r})$ at points that do **not** lie on the surface (which is just about all of them!) have **no effect** on the integration. The integral **only** depends on the value of the vector field as we move over surface S—we denote these values as $\mathbf{A}(\bar{r}_s)$.

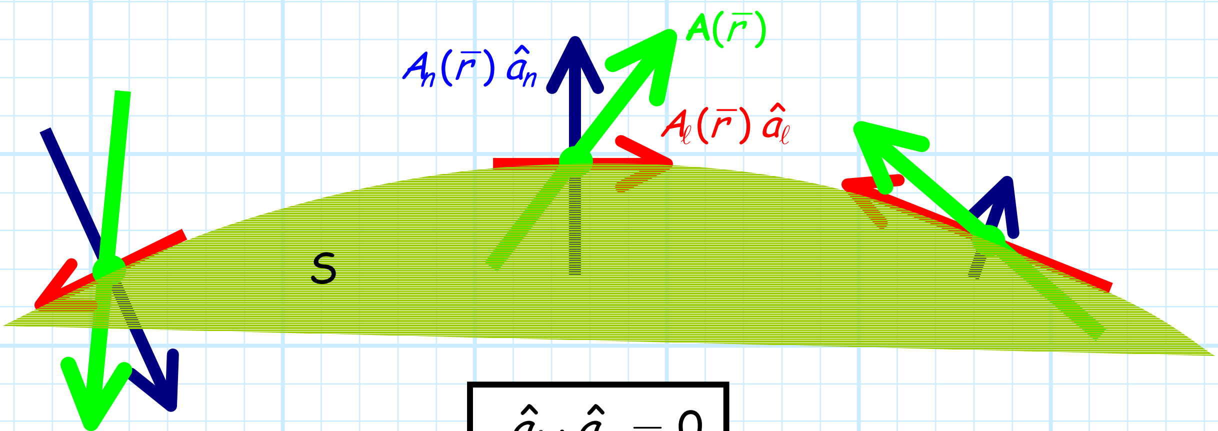
Moreover, the surface integral depends on **only one component** of $\mathbf{A}(\bar{r}_s)$!

Q: *On just what component of* $\mathbf{A}(\bar{r}_s)$ ***does*** *the integral depend?*

A: Look at the integrand $\mathbf{A}(\bar{r}_s)\cdot \overline{ds}$ --we see it involves the **dot** product! Thus, we find that the scalar integrand is simply the **scalar projection** of $\mathbf{A}(\bar{r}_s)$ onto the differential vector $\overline{ds}$. As a result, the integrand depends **only** the component of $\mathbf{A}(\bar{r}_s)$ that lies in the direction of $\overline{ds}$ --and $\overline{ds}$ **always** points in the direction orthogonal to surface S!

To help see this, first note that every vector $\mathbf{A}(\bar{r}_s)$ can be written in terms of a component tangential to the surface (i.e, $A_\ell(\bar{r}_s)\,\hat{a}_\ell$), and a component that is **normal** (i.e., orthogonal) to the surface (i.e, $A_n(\bar{r}_s)\,\hat{a}_n$):

$$\mathbf{A}(\bar{r}_s) = A_\ell(\bar{r}_s)\,\hat{a}_\ell + A_n(\bar{r}_s)\,\hat{a}_n$$



$$\hat{a}_\ell \cdot \hat{a}_n = 0$$

We note that the differential surface vector $\overline{ds}$ can be written in terms of its magnitude ($|\overline{ds}|$) and direction ($\hat{a}_n$) as:

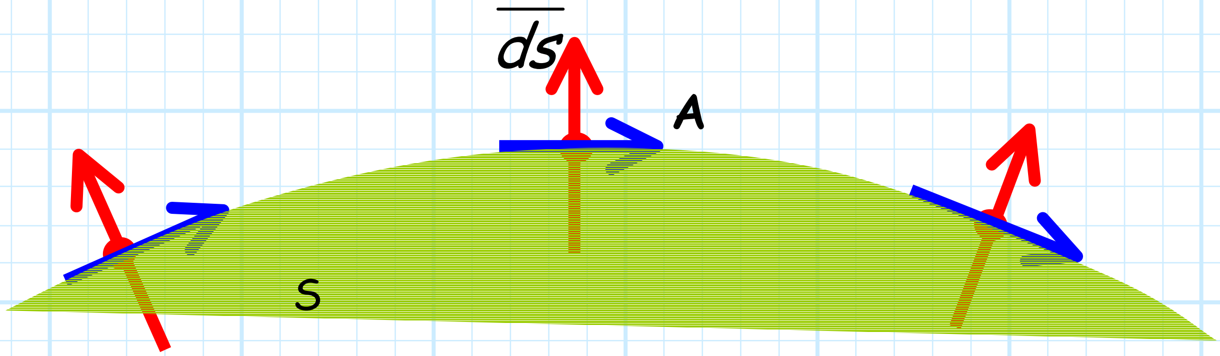
$$\overline{ds} = \hat{a}_n\,|\overline{ds}|$$

For example, for $\overline{ds_r} = \hat{a}_r\, r^2 \sin\theta\, d\theta d\phi$, we can say $|\overline{ds_r}| = r^2 \sin\theta\, d\theta\, d\phi$ and $\hat{a}_n = \hat{a}_r$.

As a result we can write:

$$\begin{aligned}\iint_S \mathbf{A}(\bar{r})\cdot\overline{ds} &= \iint_S \left[A_\ell(\bar{r})\,\hat{a}_\ell + A_n(\bar{r})\,\hat{a}_n\right]\cdot\overline{ds}\\ &= \iint_S \left[A_\ell(\bar{r})\,\hat{a}_\ell + A_n(\bar{r})\,\hat{a}_n\right]\cdot\,\hat{a}_n\left|\overline{ds}\right|\\ &= \iint_S \left[A_\ell(\bar{r})\,\hat{a}_\ell\cdot\hat{a}_n + A_n(\bar{r})\,\hat{a}_n\cdot\hat{a}_n\right]\left|\overline{ds}\right|\\ &= \iint_S A_n(\bar{r})\left|\overline{ds}\right|\end{aligned}$$

Note if vector field $\mathbf{A}(\bar{r})$ is **tangential** to the surface at every point, then the resulting surface integral will be **zero**.



Although S represents **any** surface, no matter how **complex** or **convoluted**, we will study only **basic** surfaces. In other words, $\overline{ds}$ will correspond to one of the differential surface vectors from Cartesian, cylindrical, or spherical coordinate systems.

The Surface S

In this class, we will limit ourselves to studying only those surfaces that are formed when we change the location of a point by varying **two** coordinate parameters. In other words, the other coordinate parameters will remain **fixed**.

Mathematically, therefore, a **surface** is described by:

1 equality (e.g., $x=2$ or $r=3$)

AND

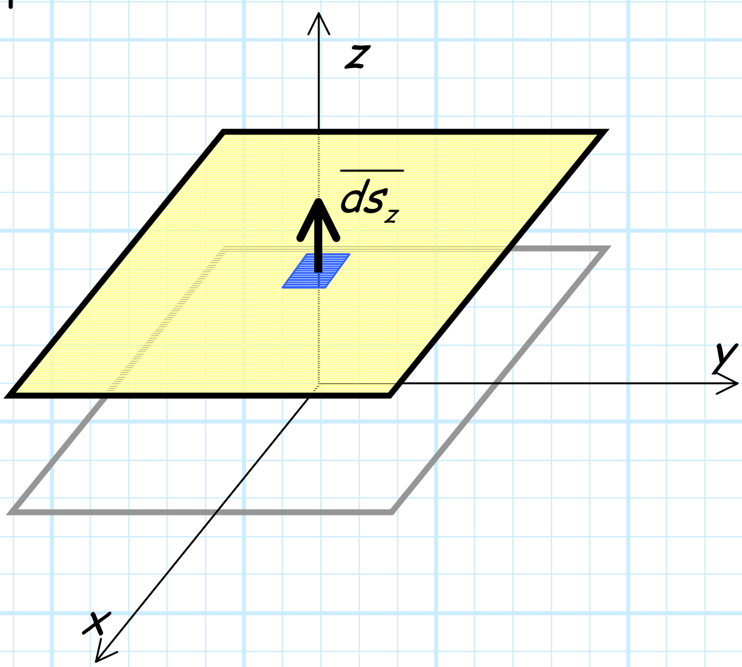
2 inequalities (e.g., $-1 < y < 5$ and $-2 < z < 7$, or $0 < \theta < \pi/2$ and $0 < \phi < \pi$)

Likewise, we will need to **explicitly** determine the **differential surface vector** $\overline{ds}$ for each contour.

We will be able to describe a surface for **each** of the coordinate values we have studied in this class!

Cartesian Coordinate Surfaces

The **single** equation $z = 3$ specifies **all** points $P(x,y,z)$ with a coordinate value $z = 3$. These points form a plane that is **parallel** to the x-y plane.



* As we move across this plane, the coordinate values of x and y **will** vary. Thus, the size of this **rectangular** plane is defined by **two inequalities** --
$c_{x1} \le x \le c_{x2}$ and $c_{y1} \le y \le c_{y2}$.

* Note the **differential surface vector** $\overline{ds_z}$ (or $-\overline{ds_z}$) is **orthogonal** to every point on this plane.

* Similarly, the equations $y = -2$ or $x = 6$ describe **planes** orthogonal to the x-z plane and the y-z plane, respectively. Likewise, the differential surface vectors $\overline{ds_y}$ and $\overline{ds_x}$ are orthogonal to each point on **these** planes.

Summarizing the **Cartesian surfaces**:

1. Flat **plane** parallel to the *y-z* plane.

$$x = c_x \qquad c_{y1} \leq y \leq c_{y2} \qquad c_{z1} \leq z \leq c_{z2}$$

$$\overline{ds} = \pm \overline{ds_x} = \pm \hat{a}_x \, dy \, dz$$

2. Flat **plane** parallel to the *x-z* plane.

$$c_{x1} \leq x \leq c_{x2} \qquad y = c_y \qquad c_{z1} \leq z \leq c_{z2}$$

$$\overline{ds} = \pm \overline{ds_y} = \pm \hat{a}_y \, dz \, dx$$

3. Flat **plane** parallel to the *x-y* plane.

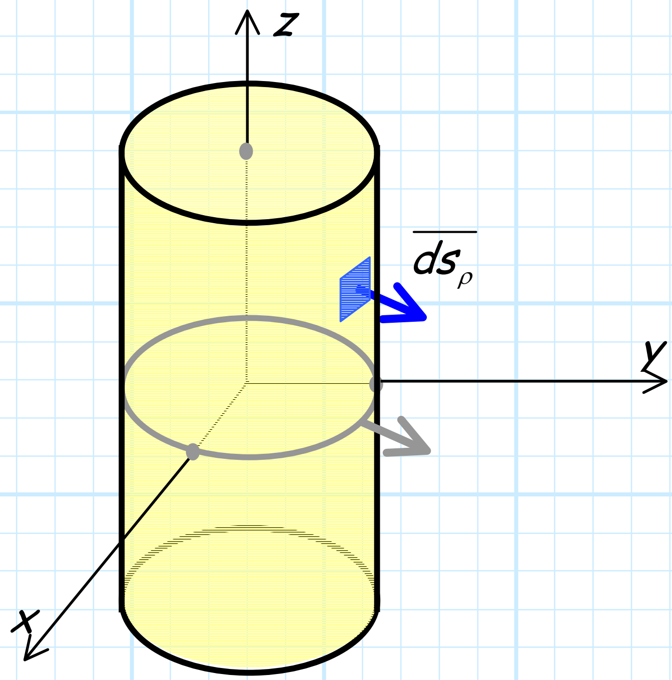
$$c_{x1} \leq x \leq c_{x2} \qquad c_{y1} \leq y \leq c_{y2} \qquad z = c_z$$

$$\overline{ds} = \pm \overline{ds_z} = \pm \hat{a}_z \, dy \, dx$$

Cylindrical Coordinate Surfaces

With **cylindrical** coordinates, we can define surfaces such as $\phi = 45^\circ$ or $\rho = 4$. These surfaces, however, are more **complex** than simply planes.

For example, the surface denoted by $\rho=4$ is formed by all points with coordinate $\rho=4$. In other words, this surface is formed by **all** points that are a distance of 4 units from the *z*-axis—a **cylinder** !

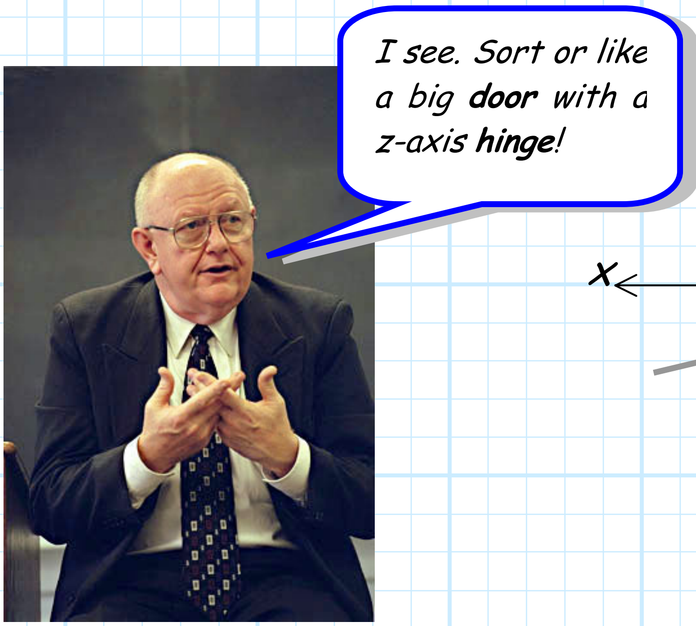


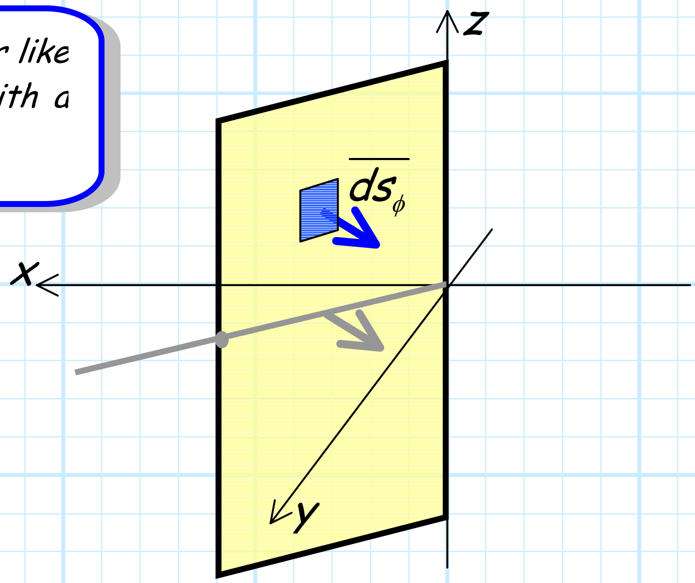
* As we move across this cylinder, the coordinate values of ϕ and *z* **will** vary. Thus, the size of this cylinder is defined by **two inequalities**--$c_{\phi 1} \le \phi \le c_{\phi 2}$ and $c_{z1} \le z \le c_{z2}$.

* Note a cylinder that **completely surrounds** the *z*-axis is described by the inequality $0 \le \phi \le 2\pi$. However, the cylinder does **not** have to be complete! For example, the inequality $0 \le \phi \le \pi$ defines a **half**-cylinder,

* We note the differential surface vector $\overline{ds}_\rho$ (or $-\overline{ds}_\rho$) is orthogonal to this surface at **all** points.

Another surface is defined by the equation $\phi = 45^\circ$. This surface is formed only from points with coordinate value $\phi = 45^\circ$. The surface is a **half-plane** that extends outward from the *z*-axis.



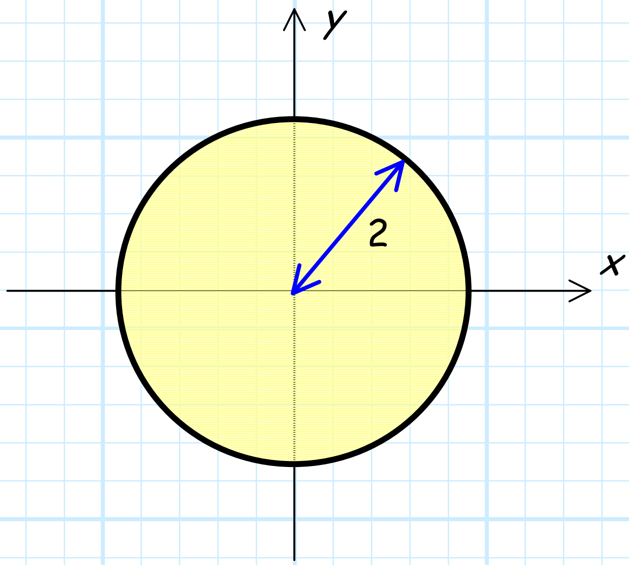


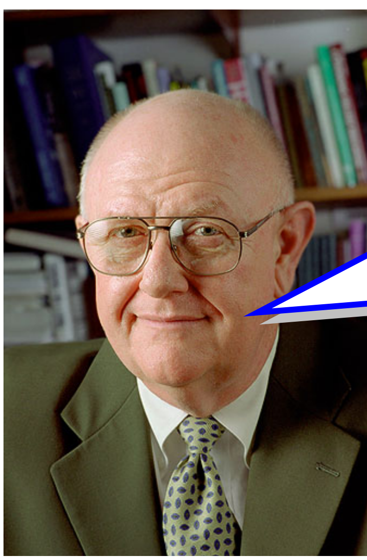
Note the differential surface vector $\overline{ds}_\phi$ is **orthogonal** to this surface at every point.

The **final cylindrical surface** that we will consider the type formed by the equality *z* = 2. We know that this forms a **flat plane** that is parallel to the *x-y* plane.

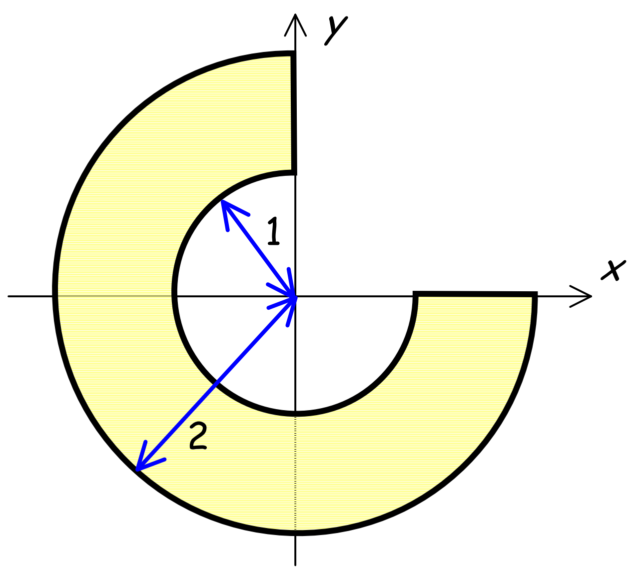
* Using the inequalities of **Cartesian** coordinates, this flat plane is rectangular in shape. However, using **cylindrical** coordinates inequalities, this plane will be shaped like a **ring** or a **disk**.

* For example, the surface $z = 0,\ 0 \le \rho \le 2,\ 0 \le \phi \le 2\pi$ describes a circular disk of radius 2, lying **on** the x-y plane, and centered at the **z**-axis:





Now let's see if ***you've*** *been paying attention! Determine the two* ***inequalities*** *that define this flat surface.*



$z = 0$

$\leq \rho \leq$

$\leq \phi \leq$

Summarizing our **cylindrical surface** results:

1. Circular **cylinder** centered around the *z*-axis.

$$\rho = c_\rho \qquad c_{\phi 1} \leq \phi \leq c_{\phi 2} \qquad c_{z1} \leq z \leq c_{z2}$$

$$\overline{ds} = \pm \overline{ds_\rho} = \pm \hat{a}_p \, \rho \, d\phi \, dz$$

2. "Vertical" **half-plane** extending from the *z*-axis.

$$c_{\rho 1} \leq \rho \leq c_{\rho 2} \qquad \phi = c_\phi \qquad c_{z1} \leq z \leq c_{z2}$$

$$\overline{ds} = \pm \overline{ds_\phi} = \pm \hat{a}_\phi \, dz \, d\rho$$

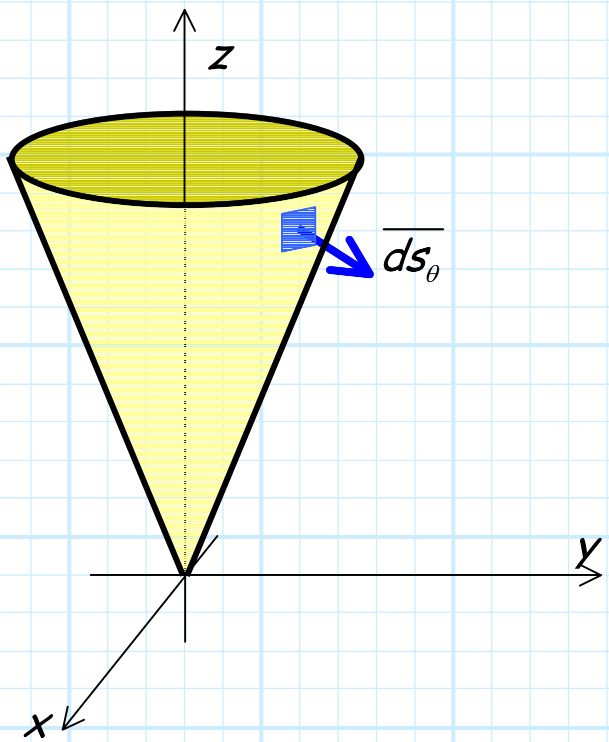
3. Flat **plane** parallel to the *x-y* plane.

$$c_{\rho 1} \leq \rho \leq c_{\rho 2} \qquad c_{\phi 1} \leq \phi \leq c_{\phi 2} \qquad z = c_z$$

$$\overline{ds} = \pm \overline{ds_z} = \pm \hat{a}_z \, \rho \, d\phi \, d\rho$$

Spherical Coordinate Surfaces

The surface defined by $\theta = 30^\circ$ is formed only from points with coordinate $\theta = 30^\circ$. This surface is a **cone**! The apex of the cone is centered at the origin, and its axis of rotation is the z-axis.



* Note that the differential surface vector $\overline{ds_\theta}$ is **normal** to this surface at every point.

* Just like a cylinder, a **complete** cone is defined by the inequality $0 \le \phi \le 2\pi$. Alternatively, for example, the equation $\pi \le \phi \le 3\pi/2$ defines a **quarter** cone.

Say instead our equality equation is $r = 3$. This defines a surface formed from all points a distance of 3 units from the origin—a **sphere** of radius 3 !

* This sphere is **centered** at the origin.

* The differential surface vector $\overline{ds_r}$ is normal to this sphere at all points on the surface.

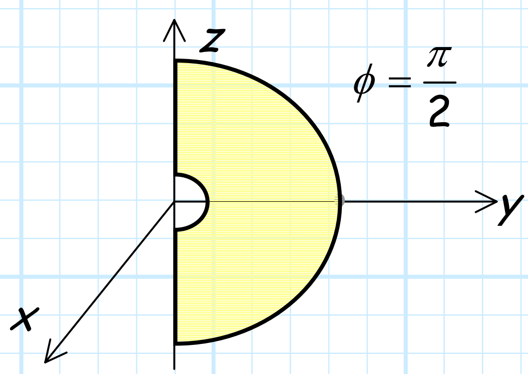
* If we wish to define a **complete** sphere, our inequalities must be:

$$0 \leq \theta < \pi \qquad \text{and} \qquad 0 \leq \phi < 2\pi$$

otherwise, we will be defining some **subsection** of a spherical surface (e.g., the "Northern Hemisphere".).

Finally, we know that the equation $\phi = 45^\circ$ defines a vertical **half-plane**, extending from the z-axis.

However, using **spherical** inequalities, this vertical plane will be in the shape of a **semi-circle** (or some section thereof), as opposed to rectangular (with cylindrical inequalities).



Summarizing the **spherical surfaces**:

1. **Sphere** centered at the origin.

$$r = c_r \qquad c_{\theta 1} \leq \theta \leq c_{\theta 2} \qquad c_{\phi 1} \leq \phi \leq c_{\phi 2}$$

$$\overline{ds} = \pm \overline{ds_r} = \pm \hat{a}_r \, r^2 \sin\theta \, d\theta \, d\phi$$

2. A **cone** with apex at the origin and aligned with the *z*-axis.

$$c_{r1} \leq r \leq c_{r2} \qquad \theta = c_\theta \qquad c_{\phi 1} \leq \phi \leq c_{\phi 2}$$

$$\overline{ds} = \pm \overline{ds_\theta} = \pm \hat{a}_\theta \, r \sin\theta \, d\phi \, dr$$

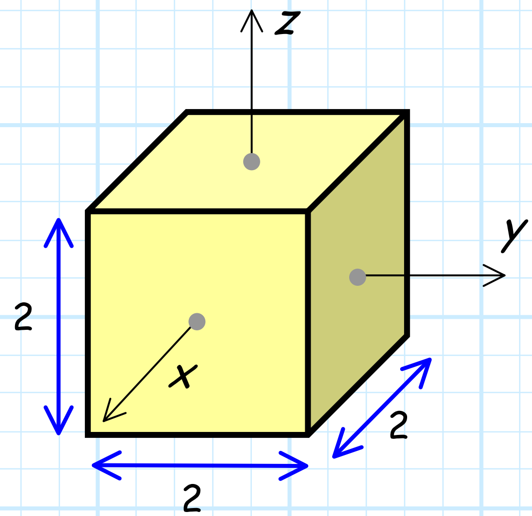
3. "Vertical" **half-plane** extending **from** the *z*-axis.

$$c_{r1} \leq r \leq c_{r2} \qquad c_{\theta 1} \leq \theta \leq c_{\theta 2} \qquad \phi = c_\phi$$

$$\overline{ds} = \pm \overline{ds_\phi} = \pm \hat{a}_\phi \, r \, dr \, d\theta$$

Integrals with Complex Surfaces

Similar to contours, we can form complex surfaces by combining any of the **seven** simple surfaces that can easily be formed with Cartesian, cylindrical or spherical coordinates. For example, we can define **6 planes** to form the surface of a **cube** centered at the origin:



The cube surface S is thus described as the sum of the **six** sides:

$$S = S_1 + S_2 + S_3 + S_4 + S_5 + S_6$$

Therefore, a surface integration over S can be evaluated as:

$$\iint_S \mathbf{A}(\bar{r}_s)\cdot\overline{ds} = \iint_{S_1} \mathbf{A}(\bar{r}_s)\cdot\overline{ds} + \iint_{S_2} \mathbf{A}(\bar{r}_s)\cdot\overline{ds} + \iint_{S_3} \mathbf{A}(\bar{r}_s)\cdot\overline{ds}$$

$$+\iint_{S_4} \mathbf{A}(\bar{r}_s)\cdot\overline{ds} + \iint_{S_5} \mathbf{A}(\bar{r}_s)\cdot\overline{ds} + \iint_{S_6} \mathbf{A}(\bar{r}_s)\cdot\overline{ds}$$

This is a great example for considering the **direction** of differential surface vector $\overline{ds}$.

Recall there are **two** differential surface vectors that are orthogonal to every surface: the first is simply the **opposite** of the second.

For example, if we were performing a surface integration over the top surface of this cube (i.e., z=1 plane) , we would **typically** use $\overline{ds} = \overline{ds}_z = \hat{a}_z \, dx \, dy$.

However, we could **also** use the differential surface vector $\overline{ds} = -\overline{ds}_z = -\hat{a}_z \, dx \, dy$!

> **Q:** *How would the results of the two integrations* ***differ?***
>
> **A:** By a factor of **−1** !!

We find that a surface integration using $\overline{ds}$ is related to the surface integration using $-\overline{ds}$ as:

$$\iint_S \mathbf{A}(\overline{r}_s) \cdot (-\overline{ds}) = -\iint_S \mathbf{A}(\overline{r}_s) \cdot \overline{ds}$$

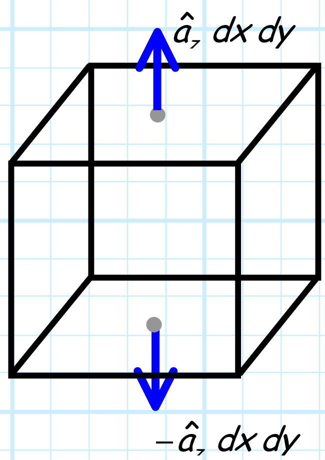
The surface of a cube is an example of a **closed surface.** A closed surface is a surface that **completely surrounds** some volume. You cannot get from **one side** of a closed surface to the **other** side without **passing through** the surface.

In other words, if your **beverage** is surrounded by a closed surface, better go get your **can opener**!

In electromagnetics, we **often** define $\overline{ds}$ as the direction **pointing outward** from a **closed** surface.

So, for example, the differential surface vector for the **top** surface (z=1) would be:

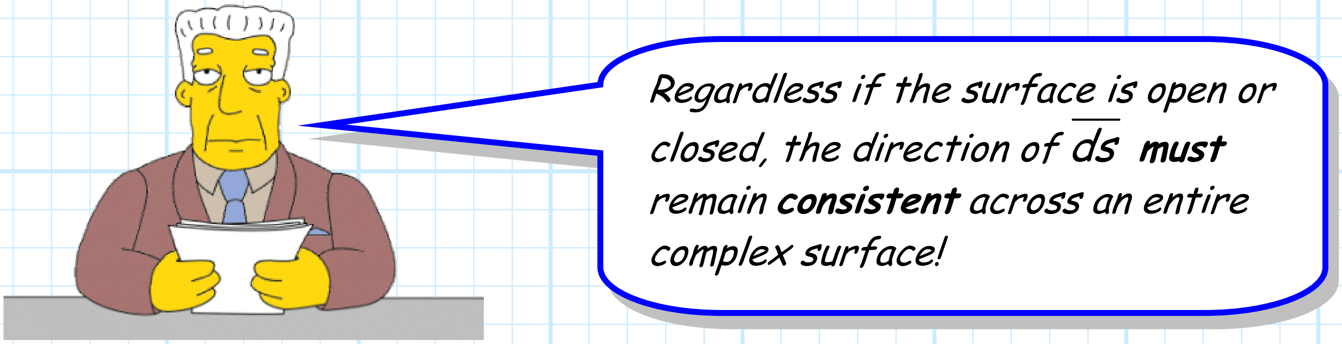
$$\overline{ds} = \overline{ds_z} = \hat{a}_z \, dx \, dy \, ,$$



while on the **bottom** (z=-1) we would use :

$$\overline{ds} = -\overline{ds_z} = -\hat{a}_z \, dx \, dy$$

Similarly, we would use differential line vectors of **opposite** directions for each of the pair of side surfaces (left and right), as well as for the front and back surfaces.



Steps for Analyzing Surface Integrals

We wish to **evaluate** an integral of the form:

$$\iint_S \mathbf{A}(\bar{r}_s) \cdot \overline{ds}$$

To successfully accomplish this, simply follow **these** steps:

Step 1: Determine the 1 equality, 2 inequalities, and $\overline{ds}$ for the surface S (be careful of direction!).

Step 2: **Evaluate** the dot product $\mathbf{A}(\bar{r}_s) \cdot \overline{ds}$.

Step 3: Write the resulting scalar field using the **same** coordinate system as surface S.

Step 4: Evaluate the scalar field using the coordinate **equality** that described surface S.

Step 5: Determine the **limits of integration** from the **inequalities** that describe surface S.

Step 6: Integrate the remaining function of **two** coordinate variables.

Example: The Surface Integral

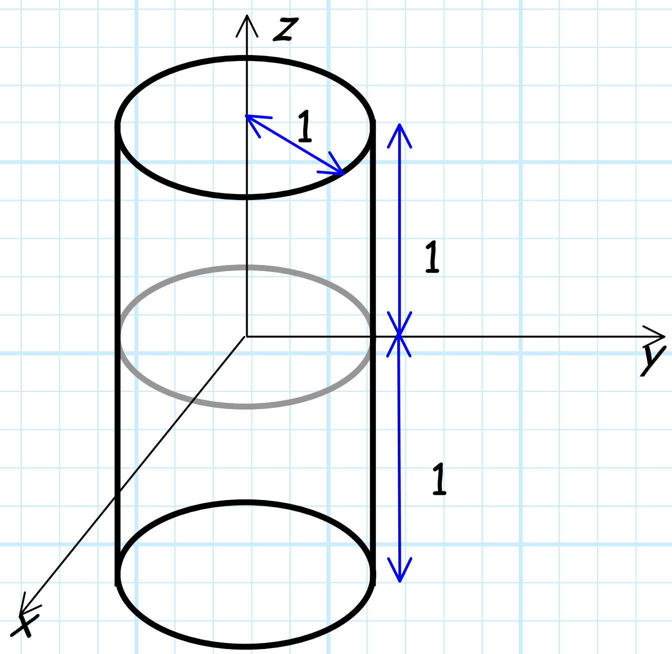
Consider the vector field:

$$\mathbf{A}(\bar{r}) = x\ \hat{a}_x$$

Say we wish to **evaluate** the surface integral:

$$\iint_S \mathbf{A}(\bar{r}_s) \cdot \overline{ds}$$

where S is a **cylinder** whose axis is aligned with the z-axis and is centered at the origin. This cylinder has a **radius** of 1 unit, and extends 1 unit below the x-y plane and one unit above the x-y plane. In other words, the cylinder has a **height** of 2 units.



This is a **complex**, **closed** surface. We will define the **top** of the cylinder as surface S_1, the **side** as S_2, and the **bottom** as S_3. The surface integral will therefore be evaluated as:

$$\iint_S \mathbf{A}(\bar{r}_s)\cdot\overline{ds} = \iint_{S_1} \mathbf{A}(\bar{r}_s)\cdot\overline{ds_1} + \iint_{S_2} \mathbf{A}(\bar{r}_s)\cdot\overline{ds_2} + \iint_{S_3} \mathbf{A}(\bar{r}_s)\cdot\overline{ds_3}$$

Step 1: Determine $\overline{ds}$ for the surface S.

Let's define $\overline{ds}$ as pointing in the direction outward from the closed surface.

S_1 is a **flat plane** parallel to the *x-y* plane, defined as:

$$0 \le \rho \le 1 \quad 0 \le \phi \le 2\pi \quad z = 1$$

and whose outward pointing $\overline{ds}$ is:

$$\overline{ds_1} = \overline{ds_z} = \hat{a}_z \ \rho\, d\rho\, d\phi$$

S_2 is a **circular cylinder** centered on the *z*- axis, defined as:

$$\rho = 1 \quad 0 \le \phi \le 2\pi \quad -1 \le z \le 1$$

and whose outward pointing $\overline{ds}$ is:

$$\overline{ds_2} = \overline{ds_\rho} = \hat{a}_\rho \ \rho\, dz\, d\phi$$

S_3 is a **flat plane** parallel to the *x-y* plane, defined as:

$$0 \le \rho \le 1 \quad 0 \le \phi \le 2\pi \quad z = -1$$

and whose outward pointing $\overline{ds}$ is:

$$\overline{ds_3} = -\overline{ds_z} = -\hat{a}_z \; p\, dp\, d\phi$$

Step 2: Evaluate the dot product $\mathbf{A}(\bar{r}_s) \cdot \overline{ds}$.

$$\begin{aligned} \mathbf{A}(\bar{r}_s) \cdot \overline{ds_1} &= x \; \hat{a}_x \cdot \hat{a}_z \; \rho \, d\rho \, d\phi \\ &= x\,(0)\, \rho \, d\rho \, d\phi \\ &= 0 \end{aligned}$$

$$\begin{aligned} \mathbf{A}(\bar{r}_s) \cdot \overline{ds_2} &= x \; \hat{a}_x \cdot \hat{a}_p \; \rho \, dz \, d\phi \\ &= x\,(\cos\phi)\, \rho \, dz \, d\phi \end{aligned}$$

$$\begin{aligned} \mathbf{A}(\bar{r}_s) \cdot \overline{ds_3} &= -x \; \hat{a}_x \cdot \hat{a}_z \; \rho \, d\rho \, d\phi \\ &= -x\,(0)\, \rho \, d\rho \, d\phi \\ &= 0 \end{aligned}$$

Look! Vector field $A(\bar{r})$ is **tangential** to surface S_1 and S_3 for all points on surface S_1 and S_3! Therefore:

$$\begin{aligned} \iint_S \mathbf{A}(\bar{r}_s) \cdot \overline{ds} &= \iint_{S_1} \mathbf{A}(\bar{r}_s) \cdot \overline{ds_1} + \iint_{S_2} \mathbf{A}(\bar{r}_s) \cdot \overline{ds_2} + \iint_{S_3} \mathbf{A}(\bar{r}_s) \cdot \overline{ds_3} \\ &= 0 + \iint_{S_2} \mathbf{A}(\bar{r}_s) \cdot \overline{ds_2} + 0 \\ &= \iint_{S_2} \mathbf{A}(\bar{r}_s) \cdot \overline{ds_2} \end{aligned}$$

Step 3: Write the resulting scalar field using the same coordinate system as $\overline{ds}$.

The differential vector $\overline{ds_\rho}$ is expressed in **cylindrical** coordinates, therefore we must write the **scalar** integrand using cylindrical coordinates.

We know that:

$$x = \rho\cos\phi$$

Therefore:

$$\begin{aligned}\mathbf{A}(\bar{r}_s)\cdot\overline{ds_2} &= x(\cos\phi)\,\rho\, dz\, d\phi \\ &= \rho\cos\phi(\cos\phi)\,\rho\, dz\, d\phi \\ &= \rho^2\cos^2\phi\ dz\, d\phi\end{aligned}$$

Step 4: Evaluate the scalar field using the coordinate **equality** that described surface S.

Every point on S_2 has the coordinate value $\rho = 1$. Therefore:

$$\begin{aligned}\mathbf{A}(\bar{r}_s)\cdot\overline{ds_2} &= \rho^2\cos^2\phi\ dz\, d\phi \\ &= 1^2\cos^2\phi\ dz\, d\phi \\ &= \cos^2\phi\ dz\, d\phi\end{aligned}$$

Step 5: Determine the **limits of integration** from the **inequalities** that describe surface S.

For S_2 we know that $0 \le \phi \le 2\pi \quad -1 \le z \le 1$.

Therefore:

$$\iint_S \mathbf{A}(\bar{r}_s)\cdot \overline{ds} = \iint_{S_2} \mathbf{A}(\bar{r}_s)\cdot \overline{ds_2} = \int_0^{2\pi}\int_{-1}^{1} \cos^2\phi \; dz \, d\phi$$

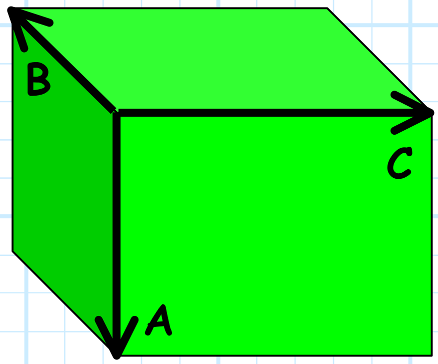
Step 6: Integrate the remaining function of **two** coordinate variables.

Using **all** the results determined above, the surface integral becomes:

$$\begin{aligned}\iint_S \mathbf{A}(\bar{r}_s)\cdot \overline{ds} &= \int_0^{2\pi}\int_{-1}^{1} \cos^2\phi \; dz \, d\phi \\ &= \int_0^{2\pi} \cos^2\phi \; d\phi \int_{-1}^{1} dz \\ &= (\pi - 0)(1-(-1)) \\ &= \mathbf{2\pi}\end{aligned}$$

The Differential Volume Element

Consider a **rectangular cube**, whose **three** sides can be defined by **three** directed distances, say **A**, **B**, and **C**.



It is evident that the lengths of each side of the rectangular cube are $|\mathbf{A}|$, $|\mathbf{B}|$, and $|\mathbf{C}|$, such that the **volume** of this rectangular cube can be expressed as:

$$V = |\mathbf{A}||\mathbf{B}||\mathbf{C}|$$

Consider now what happens if we take the **triple product** of these three vectors:

$$\mathbf{A} \cdot \mathbf{B} \times \mathbf{C} = \mathbf{A} \cdot \hat{a}_n |\mathbf{B}||\mathbf{C}| \sin\theta_{BC}$$

However, we note that $\sin\theta_{BC} = \sin 90^\circ = 1.0$, and that $\hat{a}_n = \hat{a}_A$ (i.e., vector $\mathbf{B} \times \mathbf{C}$ points in the same direction as vector **A**!).

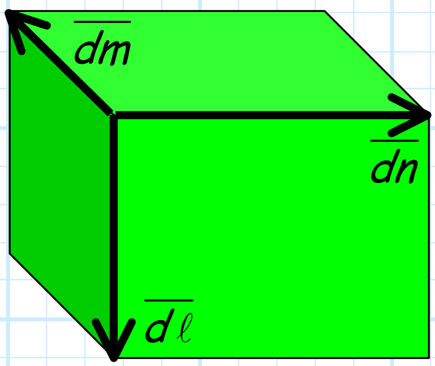
Using the fact that $\mathbf{A} = |\mathbf{A}|\,\hat{a}_A$, we then find the result:

$$\begin{aligned}\mathbf{A}\cdot\mathbf{B}\times\mathbf{C} &= \mathbf{A}\cdot\hat{a}_n\,|\mathbf{B}||\mathbf{C}|\sin\theta_{BC} \\ &= \mathbf{A}\cdot\hat{a}_A\,|\mathbf{B}||\mathbf{C}| \\ &= |\mathbf{A}|\,\hat{a}_A\cdot\hat{a}_A\,|\mathbf{B}||\mathbf{C}| \\ &= |\mathbf{A}||\mathbf{B}||\mathbf{C}|\end{aligned}$$

Look what this means, the **volume** of a cube can be expressed in terms of the **triple product**!

$$V = \mathbf{A}\cdot\mathbf{B}\times\mathbf{C} = |\mathbf{A}||\mathbf{B}||\mathbf{C}|$$

Consider now a rectangular volume formed by three orthogonal **line vectors** (e.g., $\overline{dx}$, $\overline{dy}$, $\overline{dz}$ or $\overline{d\rho}$, $\overline{d\phi}$, $\overline{dz}$).



The result is a differential volume, given as:

$$dv = \overline{d\ell}\cdot\overline{dm}\times\overline{dn}$$

For example, for the **Cartesian** coordinate system:

$$
\begin{aligned}
dv &= \overline{dx} \cdot \overline{dy} \times \overline{dz} \\
&= dx\ dy\ dz
\end{aligned}
$$

and for the **cylindrical** coordinate system:

$$
\begin{aligned}
dv &= \overline{d\rho} \cdot \overline{d\phi} \times \overline{dz} \\
&= \rho\ d\rho\ d\phi\ dz
\end{aligned}
$$

and also for the **spherical** coordinate system:

$$
\begin{aligned}
dv &= \overline{dr} \cdot \overline{d\theta} \times \overline{d\phi} \\
&= r^2 \sin\theta\ dr\ d\phi\ d\theta
\end{aligned}
$$

The Volume V

As we might expect from out knowledge about how to specify a **point** P (3 equalities), a **contour** C (2 equalities and 1 inequality), and a **surface** S (1 equality and 2 inequalities), a **volume** V is defined by **3 inequalities**.

Cartesian

The inequalities:

$$c_{x1} \leq x \leq c_{x2} \qquad c_{y1} \leq y \leq c_{y2} \qquad c_{z1} \leq z \leq c_{z2}$$

define a **rectangular volume**, whose sides are parallel to the x-y, y-z, and x-z planes.

The differential volume dv used for constructing this Cartesian volume is:

$$dv = dx \; dy \; dz$$

Cylindrical

The inequalities:

$$c_{\rho 1} \leq \rho \leq c_{\rho 2} \qquad c_{\phi 1} \leq \phi \leq c_{\phi 2} \qquad c_{z1} \leq z \leq c_{z2}$$

defines a **cylinder**, or some **subsection** thereof (e.g. a **tube**!).

The differential volume dv is used for constructing this cylindrical volume is:

$$dv = \rho \; d\rho \; d\phi \; dz$$

Spherical

The equations:

$$c_{r1} \le r \le c_{r2} \qquad c_{\theta 1} \le \theta \le c_{\theta 2} \qquad c_{\phi 1} \le \phi \le c_{\phi 2}$$

defines a **sphere**, or some subsection thereof (e.g., an "**orange slice**" !).

The differential volume dv used for constructing this spherical volume is:

$$dv = r^2 \sin\theta \, dr \, d\theta \, d\phi$$

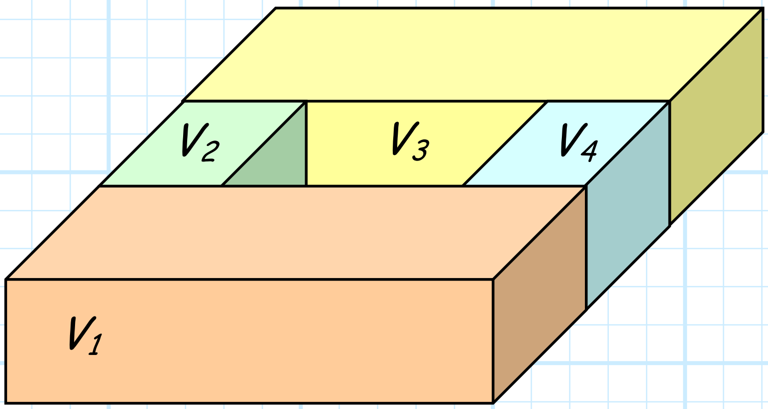
* Note that the three inequalities become **the limits of integration** for a volume integral. For example, integrating over a spherical volume would result in an integral of the form:

$$\iiint_V g(\bar{r}) \, dv = \int_{c_{\phi 1}}^{c_{\phi 2}} \int_{c_{\theta 1}}^{c_{\theta 2}} \int_{c_{r1}}^{c_{r2}} g(\bar{r}) \, r^2 \sin\theta \, dr \, d\theta \, d\phi$$

For this example, **if** the scalar field $g(\bar{r})$ is **not** expressed in terms of **spherical** coordinates, it must first be **transformed** before the integral can be explicitly **evaluated**.

* Note also that we can construct **complex volumes** by combining the simple volumes discussed here.

$$V = V_1 + V_2 + V_3 + V_4$$



Example: The Volume Integral

Let's evaluate the **volume** integral:

$$\iiint_V g(\bar{r})\, dv$$

where $g(\bar{r}) = 1$ and the volume V is a **sphere** with radius R.

In other words, the volume V is described as:

$$0 \leq r \leq R$$

$$0 \leq \theta \leq \pi$$

$$0 \leq \phi \leq 2\pi$$

And thus we use for the **differential** volume *dv:*

$$dv = \overline{dr} \cdot \overline{d\theta} \times \overline{d\phi} = r^2 \sin\theta \; dr \, d\theta \, d\phi$$

Therefore:

$$\iiint_V g(\bar{r})\, dv = \int_0^{2\pi}\int_0^{\pi}\int_0^{R} r^2 \sin\theta \, dr\, d\theta\, d\phi$$

$$= \int_0^{2\pi} d\phi \int_0^{\pi} \sin\theta d\theta \int_0^{R} r^2 dr$$

$$= 2\pi(2)\frac{R^3}{3}$$

$$= \frac{4}{3}\pi R^3$$

Hey look! The answer is the **volume** (e.g., in m^3) of a **sphere**!

Now, this result provided the numeric volume of V **only** because $g(\bar{r}) = 1$. We find that the total volume of **any** space V can be determined this way:

$$\text{Volume of } V = \iiint_V (1)\, dv$$

Typically though, we find that $g(\bar{r}) \neq 1$, and thus the volume integral does **not** provide the numeric volume of space V.

Q: *So what's the volume integral even good for?*

A: Generally speaking, the scalar function $g(\bar{r})$ will be a density function, with units of **things/unit volume**. Integrating $g(\bar{r})$ with the volume integral provides us the **number of things** within the space V!

For example, let's say $g(\bar{r})$ describes the **density** of a big **swarm of insects**, using units of ***insects/m³*** (i.e., insects are the **things**). Note that $g(\bar{r})$ must indeed a **function** of position, as the density of insects changes at different locations throughout the swarm.

Now say we want to know the total number of insects within the swarm, which occupies some space *V*. We can determine this by simply applying the volume integral!

$$\text{number of insects in swarm } = \iiint_V g(\bar{r})\, dv$$

where space *V* completely encloses the insect swarm.